Analytic Solution for the Nucleolus of a Three-Player Cooperative Game[1]

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Abstract

The nucleolus solution for cooperative games in characteristic function form is usually computed numerically by solving a sequence of linear programming (LP) problems, or by solving a single, but very large-scale, LP problem. This paper proposes an algebraic method to compute the nucleolus solution analytically (i.e., in closed-form) for a three-player cooperative game in characteristic function form. We first consider cooperative games with empty core and derive a formula to compute the nucleolus solution. Next, we examine cooperative games with non-empty core and calculate the nucleolus solution analytically for five possible cases arising from the relationship among the value functions of different coalitions.

Key words: Three-player cooperative game in characteristic function form, nucleolus, linear programming.

1 Introduction

Cooperative game theory studies situations involving multiple players who can cooperate and take joint actions in a coalition to increase their "wealth." The important problem of allocating the newly accrued wealth among the cooperating players in a fair manner has occupied game theorists since the 1940s. More than a dozen alternate solution concepts have been proposed to determine the allocation but only a few of these concepts have received the most attention. Von Neumann and Morgenstern [21] who were the originators of multiperson cooperative games proposed the first solution concept for such games known as the *stable set*. However, due to the theoretical and practical difficulties associated with it, the stable set concept fell out of favour. In 1953, Gillies [6] introduced the concept of *core* as the set of all undominated payoffs (i.e., imputations) to the players satisfying rationality properties. Even though the core has been found useful in studying economic markets, it does not provide a unique solution to the allocation problem. Also in 1953, Shapley [18] wrote three axioms which would capture the idea of a fair allocation of payoffs and developed a simple, analytic, expression to calculate the payoffs. *Shapley value* can be computed easily by using a formula regardless of whether or not the core is empty. However, when the core is non-empty, Shapley value may not be in the core and under some conditions the allocation scheme in terms of Shapley value may result in an unstable grand coalition.

An alternative solution concept known as the *nucleolus* was introduced by Schmeidler [17] in 1969 who proposed an allocation scheme that minimizes the "unhappiness" of the most unhappy player. Schmeidler [17] defines "unhappiness" (or, "excess") of a coalition as the difference between what the members of the coalition could get by themselves and what they are actually getting if they accept the allocations suggested by a solution. It was shown by Schmeidler [17] that if the core for a cooperative game is non-empty, then the nucleolus is always located inside the core and thus assures stability of the grand coalition. Unfortunately, unlike the Shapley value, there exists no closed-form formula for the nucleolus solution which has to be computed numerically in an iterative manner by solving a series of linear programming (LP) problems, or by solving a very large-scale LP problem (see, for example, Owen [14] and Wang [22] for textbook descriptions of these methods). The objective of this paper is to present analytic expressions to calculate the nucleolus solution directly without the need for iterative calculations that involve the solution of linear programs.

The nucleolus solution is an important concept in cooperative game theory even though it is not easy to calculate. As Maschler et al. [11, p. 336] pointed out, the nucleolus satisfies some desirable properties—e.g., it always exists uniquely in the core if the core is non-empty, and is therefore considered an important fair division scheme. As a consequence, some researchers have used this concept to analyze business and management problems; but, due to the complexity of the calculations, the nucleolus has not been extensively used to solve allocation-related problems. As an early application of the nucleolus concept, Barton [1] suggested the nucleolus solution as the mechanism to allocate joint costs among entities who share a common resource. Barton showed that using the nucleolus for this allocation problem can reduce the possibility that one or more entities may wish to withdraw from the resource-sharing arrangement.

To fairly divide the cost or payoff among multiple players, one should choose an allocation scheme that satisfies a natural monotonicity property. In the context of cost allocation, the monotonicity of a scheme means that, if the cost (payoff) incurred by each possible coalition rises, then the cost (payoff) allocation to each entity under the scheme should be increased. As Megiddo [12] proved, the nucleolus is not always monotonic which is considered as a drawback of this concept. We note that in Barton's cost allocation problem, if the cost for using the common resource increases, then the nucleolus solution may suggest a lower cost allocated to some entities, which means that the nucleolus is not monotonic in the problem analyzed by Barton [1].

It has been shown that there are other solution concepts that satisfy the monotonicity property and may be used instead of the nucleolus. For example, Young [23] proved that the Shapley value is a unique, monotonic solution, even though, as pointed out above, it may not be in the core if the core is non-empty. In [8], Grotte normalized the nucleolus (by dividing the "excess" of each coalition by the number of players in the coalition) and correspondingly, introduced the new concept "*per capita (normalized)* nucleolus" as an alternative to the original nucleolus solution. Grotte showed that the per capita nucleolus is monotonic and also always exists in the core if the core is non-empty. Thus, for some cost-sharing problems such as that in Barton [1], the per capita nucleolus may be better than the nucleolus solution; but, we note that the calculation for the per capita nucleolus could be even more complicated than that for the nucleolus. For other publications concerning the applications of the nucleolus, see, e.g., Du et al. [4], Gow and Thomas [7], and Leng and Parlar [10].

An n-player game in characteristic-function form (as originally formulated by von Neumann and Morgenstern [21, Ch. VI]) is defined by the set $N = \{1, 2, \ldots, n\}$ and a function $v(\cdot)$ which, for any subset (i.e., coalition) $S \subseteq N$ gives a number $v(S)$ called the value of S (see, also, Straffin [20, Ch. 23]). The characteristic value of the coalition S, denoted by $v(S)$, is the payoff that all players in the coalition S can jointly obtain. For a characteristic function game (N, v), let x_i represent an imputation (i.e., a payoff) for player $i = 1, 2, \ldots, n$. The nucleolus solution is defined as an n-tuple imputation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that the excess ("unhappiness") $e_S(\mathbf{x}) = v(S) - \sum_{i \in S} x_i$ of any possible coalition S cannot be lowered without increasing any other greater excess; see, Schmeidler [17]. With this definition, we find that the nucleolus of a cooperative game is a solution concept that makes the largest unhappiness of the coalitions as small as possible, or, equivalently, minimizes the worst inequity. In the sequential LP method that is based on lexicographic ordering (Maschler et al. [11]), to find the nucleolus solution we first reduce the largest excess $\max\{e_S(x)$, for all $S \subseteq N\}$ as much as possible, then decrease the second largest excess as much as possible, and continue this process until the n-tuple imputation $\mathbf{x}$ is determined.

Existing solution methods for the nucleolus either solve a series of linear programming (LP) problems or a single, but very large LP; see, Table 1. The description of the methods to find the nucleolus as summarized in Table 1 shows that most LP-based methods are iterative in nature and when they are not iterative, the resulting LP can be quite large (as in Kohlberg [9] and Owen [13]). For further discussions regarding these LP methods, see the online Appendix B, in which we

Year	Author(s)	Brief Description of Major Algorithms in the LP Method
1972	Kohlberg [9]	When the set of payoff vectors is a polytope, the nucleolus can be obtained as the solution of a single LP problem with n variables and $(2^n)!$ constraints.
1974	Owen [13]	When the set of payoff vectors is a polytope, the nucleolus can be obtained as the solution of a single LP problem with $2^{n+1}+n$ variables and 4^n+1 constraints.
1979	Maschler, Peleg and Shapley [11]	The nucleolus was characterized as the lexicographic center of a cooperative game, and it can be found by solving a series of $O(4^n)$ minimization LP problems with constraint coefficients of either -1, 0 or 1.
1981	Behringer [2]	Simplex based algorithm developed for general lexicographically extended linear maxmin problems to find the nucleolus by solving a sequence of $O(2^n)$ LP problems.
1981	Dragan [3]	Using the concept of coalition array, linear programs with only $O(n)$ rows and $O(2^n)$ columns are used to find the nucleolus solution.
1991	Sankaran [16]	Algorithm to find the nucleolus solution by solving a sequence of $O(2^n)$ LP problems. However, this method needs more constraints than in Behringer [2].
1994	Solymosi and Raghavan [19]	Algorithm to determine the nucleolus of an assignment game. In an (m,n)-person assignment game, the nucleolus is found in at most $m(m+3)/2$ steps, each one requiring at most $O(mn)$ elementary operations.
1996	Potters, Reijnierse and Ansing [15]	The nucleolus solution can be found by solving at most $n-1$ linear programs with at most 2^n-1 rows and 2^n+n-1 columns.
1997	Fromen [5]	By utilizing Behringer's algorithm [2], the number of LP problems to find the nucleolus is reduced to $O(n)$.

Table 1: A brief review of important algorithms to compute the nucleolus using the LP method.

compare the LP methods listed in Table 1.

In this paper we focus on three-player cooperative games in characteristic-function form, and present an algebraic method that determines the nucleolus analytically (i.e., using closed-form expressions) without the need for iterative algorithms. Furthermore, we limit our discussion to the case of *superadditive* and *essential* games. [In a superadditive game, $v(S \cup T) \geq v(S) + v(T)$ for any two disjoint coalitions S and T; and in an essential game, $v(123) > v(1) + v(2) + v(3)$; see, Straffin [20].] This is a reasonable limitation because if a game is not superadditive and/or essential, then the grand coalition will not be stable since the players would be better off by leaving this coalition. Thus, when a game is not superadditive and/or essential, it is unnecessary to examine the problem of fairly allocating the system-wide profit (that is, the characteristic value of grand coalition) among all players. An example of a 3-player game that is not essential is given by Maschler et al. [11] as $[v(\varnothing) \mid v(1), v(2), v(3) \mid v(12), v(13), v(23) \mid v(123)] = [0 \mid 0, 0, 0 \mid 0, 0, 10 \mid 6]$. Here, the grand coalition $\{1, 2, 3\}$ is not stable since coalition $\{2, 3\}$ can gain more if they do not join the grand coalition because $v(23) = 10 > v(123) = 6$.

Without loss of generality, and as justified in Straffin [20, Ch. 23, pp. 152–153], in our three-player superadditive and essential game the characteristic values of the empty and one-player coalitions are assumed zero, i.e., $v(\varnothing) = v(1) = v(2) = v(3) = 0$; the characteristic values of two-player coalitions are non-negative, i.e., $v(ij) \geq 0$, for $i,\ j = 1,\ 2,\ 3,\ i \neq j$; and the characteristic value of the grand coalition $\{123\}$ is positive, i.e., $v(123) > 0$. If this is not the case, then, as discussed in Maschler et al. [11] and demonstrated in Straffin [20, Ch. 23, pp. 153], we can transform any superadditive, and essential three-player game to a "*0-normalized*" game with zero characteristic values of all one-player coalitions. For an example, see the online Appendix A.

The remainder of the paper is organized as follows. In Section 2, we first derive a closed-form algebraic formula to compute the nucleolus solution for three-player characteristic-function cooperative games with empty core. Then, we investigate the computation of the nucleolus when the core of a cooperative game is non-empty, and present five closed-form formulas each arising from the relationship among the value functions of different coalitions. We use two examples to illustrate our algebraic method. In Section 3, we summarize the paper and provide some suggestions for future research.

2 Algebraic Method for Computing the Nucleolus Solution Analytically

In this section, we develop an algebraic method to compute the nucleolus of a three-player cooperative game analytically without the need for linear programming. That is, we derive explicit formulas to compute the nucleolus. We first present our analysis for the relatively simpler case of a cooperative game with empty core. This is followed by the more complicated analysis of the nucleolus computation for cooperative games with non-empty core.

Since we shall minimize the excesses of all possible coalitions to find the nucleolus solution, we first compute these excesses at an imputation $\mathbf{x}$ as follows:

$$e_i(\mathbf{x}) = v(i) - x_i = -x_i, \text{ for } i = 1, 2, 3, \tag{1}$$

$$e_{ij}(\mathbf{x}) = v(ij) - x_i - x_j = v(ij) - v(123) + x_k, \text{ for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k, \tag{2}$$

$$e_{123}(\mathbf{x}) = v(123) - x_1 - x_2 - x_3 = 0. \tag{3}$$

Note that due to the collective rationality assumption we have $e_{123}(\mathbf{x}) = 0$ in (3); that is, the payoff $v(123)$ of the grand coalition $\{123\}$ is divided to determine three players' payoffs x_1, x_2 and x_3. The collective rationality assumption is then used to find the equalities in (2).

2.1 Algebraic Method for Empty-Core Cooperative Games

We now consider a superadditive and essential cooperative game with empty core, and derive a formula for computing the nucleolus solution.

Theorem 1 If the core of a three-player cooperative game in characteristic function form is empty, then the nucleolus solution $\mathbf{y} = (y_1, y_2, y_3)$ is computed as

$$y_i = \frac{v(123) + v(ij) + v(ik) - 2v(jk)}{3}, \quad \text{for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k. \tag{4}$$

Proof. See the online Appendix C. ■

We use the formula in Theorem 1 to compute the nucleolus solution for the following cooperative game.

Example 1 Consider the following three-player superadditive and essential cooperative game in characteristic function form: $v(\varnothing) = 0$; $v(i) = 0$, for $i = 1, 2, 3$; $v(12) = 5$, $v(13) = 6$, $v(23) = 8$; $v(123) = 9$. It is easy to show that for this game the core is empty[1]. Using Theorem 1, we compute the nucleolus solution as $y_1 = \frac{1}{3}(5 + 6 + 9 - 2 \cdot 8) = \frac{4}{3}$, $y_2 = \frac{1}{3}(5 + 8 + 9 - 2 \cdot 6) = \frac{10}{3}$ and $y_3 = \frac{1}{3}(6 + 8 + 9 - 2 \cdot 5) = \frac{13}{3}$. ◀

2.2 Algebraic Method for Nonempty-Core Cooperative Games

We now derive the formulas that are used to compute the nucleolus solution for a three-player cooperative game with a non-empty core. Since the core is not empty, the nucleolus solution must be in the core (see, for example, Straffin [20, Ch. 23]), and thus, the excesses in (1) and (2) in terms of the nucleolus are non-positive, i.e., $e_j(\mathbf{y}) \leq 0$, for $j = \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$. In order to determine the nucleolus solution, we must first reduce the largest excesses to minimum and then decrease the second largest excess and other excesses. To that end, we first find the necessary and sufficient conditions under which the largest excesses are reduced to the minimum.

Lemma 1 For a three-player cooperative game with a non-empty core, the largest excesses are reduced to minimum if and only if at least one of the following conditions is satisfied:

1. With imputation $\mathbf{x} = (x_1, x_2, x_3) = \left(\frac{1}{3}v(123), \frac{1}{3}v(123), \frac{1}{3}v(123)\right)$, and, $v(123) \geq \max(3v(12), 3v(13), 3v(23))$.
2. With imputation

$$\mathbf{x} = (x_1, x_2, x_3) = \left(\frac{v(123) + v(12)}{2} - x_2, x_2, \frac{v(123) - v(12)}{2}\right), \tag{5}$$

and,

$$\max\left\{v(23), \frac{v(123) - v(12)}{2}\right\} \leq x_2 \leq \min\left\{v(12), \frac{v(123) + v(12)}{2} - v(13)\right\}. \tag{6}$$

[1] A simpler method for testing whether the core is empty or not is to solve the following linear program: min x_1 subject to $x_1 + x_2 \geq v(12)$, $x_1 + x_3 \geq v(13)$, $x_2 + x_3 \geq v(23)$, $x_1 + x_2 + x_3 = v(123)$, $x_i \geq 0$, $i = 1, 2, 3$. If the LP has no feasible solution, then the core is empty; otherwise the core is non-empty.

3. With imputation

$$\mathbf{x} = (x_1, x_2, x_3) = \left(x_1, \frac{v(123) - v(13)}{2}, \frac{v(123) + v(13)}{2} - x_1 \right), \tag{7}$$

and,

$$\max \left\{ v(12), \frac{v(123) - v(13)}{2} \right\} \le x_1 \le \min \left\{ v(13), \frac{v(123) + v(13)}{2} - v(23) \right\}. \tag{8}$$

4. With imputation

$$\mathbf{x} = (x_1, x_2, x_3) = \left(\frac{v(123) - v(23)}{2}, \frac{v(123) + v(23)}{2} - x_3, x_3 \right), \tag{9}$$

and,

$$\max \left\{ v(13), \frac{v(123) - v(23)}{2} \right\} \le x_1 \le \min \left\{ v(23), \frac{v(123) + v(23)}{2} - v(12) \right\}. \tag{10}$$

5. With imputation

$$x_i = \frac{v(123) + v(ij) + v(ik) - 2v(jk)}{3}, \text{ for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k, \tag{11}$$

and,

$$v(123) + v(jk) \le 2[v(ij) + v(ik)], \text{ for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k. \tag{12}$$

Proof. See the online Appendix D. ■

In Lemma 1 we have derived the necessary and sufficient conditions under which the largest excesses are minimized. In order to find the nucleolus solution, we need to reduce the second largest excess and the subsequent excesses to minimum.

Theorem 2 For a three-player, nonempty-core cooperative game in characteristic function form, the nucleolus solution $\mathbf{y} = (y_1, y_2, y_3)$ can be computed as follows:

1. If $v(123) \ge 3v(ij)$, for $i, j = 1, 2, 3$ and $i \neq j$, then $y_1 = y_2 = y_3 = \frac{1}{3}v(123)$.
2. If $v(123) \ge v(ij) + 2v(ik)$, $v(123) \ge v(ij) + 2v(jk)$ and $v(123) \le 3v(ij)$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then $y_i = y_j = \frac{1}{4}[v(123) + v(ij)]$ and $y_k = \frac{1}{2}[v(123) - v(ij)]$.
3. If $v(123) \le v(ij) + 2v(ik)$, $v(123) \ge v(ij) + 2v(jk)$ and $v(ij) \ge v(ik)$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then $y_i = \frac{1}{2}[v(ij) + v(ik)]$, $y_j = \frac{1}{2}[v(123) - v(ik)]$, and $y_k = \frac{1}{2}[v(123) - v(ij)]$.
4. If $v(123) + v(ij) \ge 2[v(ik) + v(jk)]$, $v(123) \le v(ij) + 2v(ik)$ and $v(123) \le v(ij) + 2v(jk)$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then

$$y_i = \frac{1}{4}\{v(123) + v(ij) + 2[v(ik) - v(jk)]\}, \; y_j = \frac{1}{4}\{v(123) + v(ij) + 2[v(jk) - v(ik)]\},$$
$$y_k = \frac{1}{2}[v(123) - v(ij)].$$

5. If $v(123) + v(ij) \leq 2[v(ik) + v(jk)]$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then

$$y_i = \frac{1}{3}\{v(123) + v(ij) + v(ik) - 2v(jk)\},\ y_j = \frac{1}{3}\{v(123) + v(ij) + v(jk) - 2v(ik)\},$$
$$y_k = \frac{1}{3}\{v(123) + v(ik) + v(jk) - 2v(ij)\}.$$

Proof. See the online Appendix E. ■

We observe from Theorem 2 that, as the characteristic value of the grand coalition $v(123)$ increases, the allocation to one or two players may be decreased. For example, we now consider the second case (in Theorem 2), in which $v(123) \geq v(ij) + 2v(ik)$, $v(123) \geq v(ij) + 2v(jk)$ and $v(123) \leq 3v(ij)$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. For this case, the allocation scheme suggested by the nucleolus solution is given as follow: $y_i = y_j = \frac{1}{4}[v(123) + v(ij)]$ and $y_k = \frac{1}{2}[v(123) - v(ij)]$. Since $v(ij) < v(123)$ for the superadditive and essential game, we find that $y_i = y_j \neq y_k$. If we increase $v(123)$ to a sufficiently large value $v'(123)$ so that the first case in Theorem 2 applies, then we find that the allocation scheme is changed to the following: $y_1 = y_2 = y_3 = \frac{1}{3}v'(123)$. Comparing the new allocation scheme and that obtained before we increase $v(123)$ to $v'(123)$, we find that one or two players may be worse off when the characteristic value of the grand coalition is increased. More specifically, if $v(ij) < \frac{2}{3}v'(123) - v(123)$, then $y_k = \frac{1}{2}[v(123) - v(ij)] > \frac{1}{3}v'(123)$. Because $y_i + y_j + y_k = v(123) < v'(123)$, we find that $y_i = y_j = \frac{1}{4}[v(123) + v(ij)] < \frac{1}{3}v'(123)$. It thus follows that, after the characteristic value of the grand coalition is increased from $v(123)$ to $v'(123)$, player k is worse off and players i and j are better off. We also note that, if $v(ij) > \frac{4}{3}v'(123) - v(123)$, then $y_i = y_j = \frac{1}{4}[v(123) + v(ij)] > \frac{1}{3}v'(123)$ and $y_k = \frac{1}{2}[v(123) - v(ij)] < \frac{1}{3}v'(123)$, which means that player k is better off but players i and j are worse off. This discussion demonstrates that the nucleolus is not always monotonic, as proved by Megiddo [12].

Next, we provide an example to illustrate our analytic results in the above theorem.

Example 2 We now use our algebraic method given in Theorem 2 to solve the following three-player cooperative game: $v(\varnothing) = 0$; $v(i) = 0$, for $i = 1, 2, 3$; $v(12) = 1$, $v(13) = 4$, $v(23) = 3$; $v(123) = 6$. Since the core of this game is non-empty, we use one of the formulas in Theorem 2 to find the nucleolus solution. Since $v(123) = 6 \leq v(13) + 2v(23) = 10$, $v(123) = 6 \geq v(13) + 2v(12) = 6$, $v(13) = 4 \geq v(23) = 3$, the third case (with $i = 3$, $j = 1$ and $k = 2$) in Theorem 2 is eligible to calculate the nucleolus $\mathbf{y} = (y_1, y_2, y_3)$ as $y_1 = [v(123) - v(23)]/2 = 1.5$, $y_2 = [v(123) - v(13)]/2 = 1$ and $y_3 = [v(13) + v(23)]/2 = 3.5$. ◄

We have written Maple worksheets which test the emptiness of the core (`CoreTest.mws`), and calculate the nucleolus solution when the core is empty (`Nucleolus-EmptyCore.mws`) and when it is nonempty (`Nucleolus-NonEmptyCore.mws`). These files work with Maple 10, 11 and 12, and they can be downloaded from the authors' web site at `http://www.business.mcmaster.ca/OM/parlar/files/nucleolus/`.

3 Summary and Concluding Remarks

Linear programming plays a prevalent role in computing the nucleolus solution of a cooperative game in the characteristic function form. However, this method requires the solution of a sequence of linear problems, thus making it inconvenient to use. To simplify the computations in calculating the nucleolus, we propose an algebraic method that gives the nucleolus analytically. This paper focuses on a three-player cooperative game. As discussed in Section 2.1, only a single formula is needed for computing the nucleolus solution when the core of a three-player game is empty. In Section 2.2, we derive some formulas each used under three specific conditions. Two examples are presented to illustrate our algebraic method.

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Online Appendices
"Analytic Solution for the Nucleolus of a Three-Player Cooperative Game"
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Appendix A Transformation of a Superadditive and Essential Game to a "Zero-Normalized" Game

We provide an example to show how to transform a superadditive, and essential three-player game to a "*0-normalized*" game with zero characteristic values of all one-player coalitions. Consider the game (N, v) with $[v(\varnothing) \mid v(1), v(2), v(3) \mid v(12), v(13), v(23) \mid v(123)] = [0 \mid 1, 2, 3 \mid 8, 10, 13 \mid 15]$. We can transform (N, v) to the following *strategically equivalent* game (N, v') by subtracting a suitable constant c_i from player i's payoff and (from the value of any coalition containing player i). This gives,

$$\begin{array}{r||r} v'(\varnothing) = 0 & v'(12) = v(12) - v(1) - v(2) = 5 \\ v'(1) = v(1) - 1 = 0 & v'(13) = v(13) - v(1) - v(3) = 6 \\ v'(2) = v(2) - 2 = 0 & v'(23) = v(23) - v(2) - v(3) = 8 \\ v'(3) = v(3) - 3 = 0 & v'(123) = v(123) - v(1) - v(2) - v(3) = 9. \end{array}$$

Using the analytic formula in Section 2.1, the nucleolus solution for this (empty core) game (N, v') is obtained as $\mathbf{y}' = (y_1', y_2', y_3') = (\frac{4}{3}, \frac{10}{3}, \frac{13}{3})$. The nucleolus solution for the original problem is then computed as $\mathbf{y} = (y_1, y_2, y_3) = (\frac{4}{3} + 1, \frac{10}{3} + 2, \frac{13}{3} + 3) = (\frac{7}{3}, \frac{16}{3}, \frac{22}{3})$ which satisfies the collective rationality condition $y_1 + y_2 + y_3 = v(123) = 15$.

Appendix B Sequential LP Method for Computing the Nucleolus Solution

Our brief review presented in Table 1 indicates that, as an early publication on the sequential LP method, Maschler et al. [11] used the concept of lexicographic centre to develop an LP procedure involving $O(4^n)$ LP minimization problems. This LP approach has been adopted by some textbooks (e.g., Wang [22]) as a "typical" method to calculate the nucleolus solution. However, because the LP method in [11] requires solving a large number of linear problems, later researchers have investigated methods to find more efficient LP approach for the calculation of the nucleolus solution.

We see in Table 1 that, immediately after Maschler et al. [11], Behringer [2] reduced the number of LP problems that are needed to find the nucleolus. We also find from Table 1 that, following Behringer [2], others (i.e., Dragan [3], Sankaran [16], and Solymosi and Raghavan [19]) attempted to further improve the LP method; but, they didn't find any method better than Behringer [2]. More specifically, in [3] Dragan's LP approach may need more than $O(2^n)$ linear problems even though this author claimed that only $n - 1$ linear programs can be used to find the nucleolus. In addition, the solution found by the LP approach in [3] is actually the prenucleolus rather than the nucleolus solution, as discussed by Potters et al. [15].

Sankaran [16] developed an LP approach which may require the same number of linear problems as in Behringer [2] but needs more constraints. Solymosi and Raghavan's approach in [19] is only applied to a special type of cooperative games (i.e., assignment games). Potters et al. [15] suggested an LP approach that may reduce the number of linear problems; but, this approach increases the size of each linear problem. From Table 1, we also find that Fromen [5] improved Behringer's algorithm [2] to reduce the number of linear problems without increasing each LP problem's size.

Similar to Section 2, we have written Maple worksheets to illustrate the LP method in calculating the nucleolus solutions for the cooperative games in Examples 1 and 2. Note from our above discussion that Maschler et al.'s method [11] is an early one and has been widely used by relevant textbooks (e.g., Owen [14] and Wang [22]) to solve numerical examples. Thus, we considered Maschler et al.'s method to develop the Maple worksheet `Empty-Simplex-1.mws` for the empty-core case and the Maple worksheet `Non-Empty-Simplex-2.mws` for the nonempty-core case. These files can be downloaded from the authors' web site at `http://www.business.mcmaster.ca/OM/parlar/files/nucleolus/`.

Appendix C Proof of Theorem 1

For a three-player empty-core cooperative game in characteristic form, we find from (1) that $e_i(\mathbf{x}) \leq 0$, for $i = 1, 2, 3$. However, since the core of the game is empty, at least one of $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ must be positive. Otherwise, if $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ are all equal to or less than zero, then using (2) we have $v(12) \leq x_1 + x_2$, $v(13) \leq x_1 + x_3$ and $v(23) \leq x_2 + x_3$, which implies that the core is not empty.

Therefore, the maximal excess must be one of $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. Accordingly, in order to minimize the maximal excess to find the nucleolus solution, we should change the imputation $\mathbf{x} = (x_1, x_2, x_3)$ to minimize the maximum of $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. If $e_{12}(\mathbf{x})$ is the maximum, then we reduce the value of x_3 and increase the values of x_1 and x_2; but, this raises the excesses $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. As a result, $e_{12}(\mathbf{x})$ must be equal to the maximum of $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. For example, if $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) > e_{23}(\mathbf{x})$, we can then reduce the values of x_3 and x_2 but increase the value of x_1, in order to make both $e_{12}(\mathbf{x})$ and $e_{13}(\mathbf{x})$ smaller; but this increases the excess $e_{23}(\mathbf{x})$. Thus, the process terminates only when $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ are equal. A similar argument applies to the case in which $e_{13}(\mathbf{x})$ or $e_{23}(\mathbf{x})$ is the maximum.

In conclusion, after we minimize the maximal excess, the excesses $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ must be equal and also, they must be nonnegative, i.e., $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x}) \geq 0$. We can then solve the following equations,

$$\begin{cases} v(12) - x_1 - x_2 = v(13) - x_1 - x_3, \\ v(12) - x_1 - x_2 = v(23) - x_2 - x_3, \\ v(123) = x_1 + x_2 + x_3, \end{cases}$$

and find the values of x_i, for $i = 1, 2, 3$. Because the payoffs of all three players have been chosen to minimize the maximal excess, we cannot make any change on the imputation $\mathbf{x} = (x_1, x_2, x_3)$ to reduce the other excesses $e_i(\mathbf{x})$ $(i = 1, 2, 3)$. Otherwise, the maximal excess will be increased. Thus, the nucleolus $\mathbf{y} = (y_1, y_2, y_3)$ is found as (4). ■

Appendix D Proof of Lemma 1

We show the sufficiency and necessity of these conditions.

Sufficiency. In this part, if one of five conditions is satisfied, then the largest excesses are reduced to the minimum. We begin by showing the first sufficient condition. Since $x_1 = x_2 = x_3 = \frac{1}{3}v(123)$; $v(123) \geq 3v(12)$, $v(123) \geq 3v(13)$ and $v(123) \geq 3v(23)$, we use (1) and (2) to find that

$$\begin{aligned} e_1(\mathbf{x}) &= e_2(\mathbf{x}) = e_3(\mathbf{x}) = -\frac{1}{3}v(123), \\ e_{12}(\mathbf{x}) &= v(12) - v(123) + x_3 = v(12) - \frac{2}{3}v(123) \leq -\frac{1}{3}v(123), \\ e_{13}(\mathbf{x}) &= v(13) - v(123) + x_2 = v(13) - \frac{2}{3}v(123) \leq -\frac{1}{3}v(123), \\ e_{23}(\mathbf{x}) &= v(23) - v(123) + x_1 = v(23) - \frac{2}{3}v(123) \leq -\frac{1}{3}v(123), \end{aligned}$$

which implies that at least one of the excesses $e_i(\mathbf{x})$ $(i = 1, 2, 3)$ is the largest. Next we prove that the largest excesses arrive to the minimum when $x_1 = x_2 = x_3 = v(123)/3$, that is, $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. Suppose that $e_1(\mathbf{x})$ is the largest excess and $e_2(\mathbf{x})$ and $e_3(\mathbf{x})$ are both less than $e_1(\mathbf{x})$. In order to decrease $e_1(\mathbf{x}) = -x_1$, we should increase the value of x_1. However, since $x_1 + x_2 + x_3 = v(123)$, we must reduce the value of x_2 and/or the value of x_3, thereby increasing the excess $e_2(\mathbf{x}) = -x_2$ and/or $e_3(\mathbf{x}) = -x_3$. This continues until $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. When either $e_2(\mathbf{x})$ or $e_3(\mathbf{x})$ is the largest, we can obtain the same result. Thus, we can conclude that if $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$, $v(123) \geq 3v(12)$, $v(123) \geq 3v(13)$ and $v(123) \geq 3v(23)$, then the largest excesses arrive to the minimum; thus we reach the first sufficient condition.

We then discuss the second sufficient condition. From (5) we have $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$. Recalling from (2) that $e_3(\mathbf{x}) = -x_3$ and $e_{12}(\mathbf{x}) = v(12) - v(123) + x_3$, we find that in order to reduce the excess $e_3(\mathbf{x})$, we should increase the value of x_3. However, this increases the value of $e_{12}(\mathbf{x})$. Therefore, we cannot change the imputation $\mathbf{x} = (x_1, x_2, x_3)$ to reduce both $e_3(\mathbf{x})$ and $e_{12}(\mathbf{x})$ simultaneously. Next, we show that $e_3(\mathbf{x})$ and $e_{12}(\mathbf{x})$ are two largest excesses; that is, we should prove that $e_3(\mathbf{x}) - e_1(\mathbf{x}) \geq 0$, $e_3(\mathbf{x}) - e_2(\mathbf{x}) \geq 0$, $e_3(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0$ and $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) \geq 0$.

1. From (1) we find that $e_3(\mathbf{x}) - e_1(\mathbf{x}) = -x_3 + x_1$. Using (5) we compute

$$e_3(\mathbf{x}) - e_1(\mathbf{x}) = \frac{v(123) + v(12)}{2} - x_2 - \frac{v(123) - v(12)}{2} = v(12) - x_2,$$

and we find that $e_3(\mathbf{x}) - e_1(\mathbf{x}) \geq 0$, which results from (6).

2. From (1) we find that $e_3(\mathbf{x}) - e_2(\mathbf{x}) = -x_3 + x_2$. Using (5) we compute
$$e_3(\mathbf{x}) - e_2(\mathbf{x}) = x_2 - \frac{v(123) - v(12)}{2},$$
and we find that $e_3(\mathbf{x}) - e_2(\mathbf{x}) \geq 0$ according to (6).
3. From (1) and (2) we find that $e_3(\mathbf{x}) - e_{13}(\mathbf{x}) = -v(13) + v(123) - x_3 - x_2 = -v(13) + x_1$. Using (5) we compute
$$e_3(\mathbf{x}) - e_{13}(\mathbf{x}) = -v(13) + x_1 = \frac{v(123) + v(12)}{2} - v(13) - x_2,$$
and we find that $e_3(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0$ according to (6).
4. From (1) and (2) we also find that $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) = -v(23) + v(123) - x_1 - x_3 = -v(23) + x_2$. Using (5) we compute $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) = -v(23) + x_2$ and, using (6), we find that $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) \geq 0$.

Similarly, we can show the sufficient conditions 3 and 4. Next we discuss the last sufficient condition. Using (11) we have $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x}) = [v(12) + v(13) + v(23) - 2v(123)]/3$. Next, we show that these three excesses are the largest, i.e., $e_{12}(\mathbf{x}) \geq e_i(\mathbf{x})$, $i = 1, 2, 3$. From (1) and (2) we find that $e_{12}(\mathbf{x}) - e_1(\mathbf{x}) = v(12) - v(123) + x_3 + x_1 = v(12) - x_2$. According to (11) we have $x_2 = [v(123) + v(12) + v(23) - 2v(13)]/3$, and thus compute $e_{12}(\mathbf{x}) - e_1(\mathbf{x}) = [2v(12) + 2v(13) - v(123) - v(23)]/3$. From (12) we find that $e_{12}(\mathbf{x}) - e_1(\mathbf{x}) \geq 0$, or, $e_{12}(\mathbf{x}) \geq e_1(\mathbf{x})$. We can analogously show the $e_{12}(\mathbf{x}) \geq e_2(\mathbf{x})$ and $e_{12}(\mathbf{x}) \geq e_3(\mathbf{x})$. Hence, we conclude that if the conditions (11) and (12) are satisfied, then the largest excesses are reduced to the minimum.

Necessity. In this part, if the largest excesses are reduced to the minimum, then at least one of five conditions must be satisfied. Note that each of the six excesses $e_1(\mathbf{x})$, $e_2(\mathbf{x})$, $e_3(\mathbf{x})$, $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ could be largest. Next, assuming that each of these excesses is the largest, we change the imputation $\mathbf{x} = (x_1, x_2, x_3)$ under the constraint $x_1 + x_2 + x_3 = v(123)$ until it is reduced to the minimum.

1. If $e_1(\mathbf{x})$ is the largest excess, then according to (1) we can increase the value of x_1 to reduce this excess. However, from (2) we find that increasing x_1 shall raise the excess $e_{23}(\mathbf{x})$. Furthermore, because $x_1 + x_2 + x_3 = v(123)$, we should decrease x_2 and x_3, so increasing $e_2(\mathbf{x})$ and $e_3(\mathbf{x})$ in (1). Note that the excesses $e_{12}(\mathbf{x})$ and/or $e_{13}(\mathbf{x})$ in (2) decrease when we decrease x_2 and/or x_3 to reduce the largest excess $e_1(\mathbf{x})$. Thus, the largest excess reaches the minimum when $e_1(\mathbf{x}) = e_{23}(\mathbf{x})$ or $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$.
Consider the case that $e_1(\mathbf{x}) = e_{23}(\mathbf{x})$ and they are the largest excesses. Using (1) and (2) we have the equation $-x_1 = v(23) - v(123) + x_1$ and solve it to obtain $x_1 = [v(123) - v(23)]/2$. Since $x_1 + x_2 + x_3 = v(123)$, we reach (9). In addition, since $e_1(\mathbf{x})$ is the largest excess, we

have

$$\begin{cases} e_1(\mathbf{x}) - e_2(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_3(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{12}(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0, \end{cases} \quad \text{or} \quad \begin{cases} -x_1 + x_2 \geq 0, \\ -x_1 + x_3 \geq 0, \\ x_2 \geq v(12), \\ x_3 \geq v(13), \end{cases}$$

which is equivalent to (10). Thus, the fourth condition including (9) and (10) corresponds to this case.

Next, we discuss the case that $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$ and they are the largest excesses. According to (1), we find $-x_1 = -x_2 = -x_3$ and use $x_1 + x_2 + x_3 = v(123)$ to attain the imputation $\mathbf{x} = (x_1, x_2, x_3) = (v(123)/3, v(123)/3, v(123)/3)$. Because $e_1(\mathbf{x})$ is the largest excess, we have

$$\begin{cases} e_1(\mathbf{x}) - e_{12}(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{23}(\mathbf{x}) \geq 0, \end{cases} \quad \text{or} \quad \begin{cases} x_2 \geq v(12), \\ x_3 \geq v(13), \\ x_1 \geq v(23). \end{cases}$$

Replacing x_i (for $i = 1, 2, 3$) with their solutions and simplifying the above inequalities give $v(123) \geq \max(3v(12), 3v(13), 3v(23))$. Thus, we reach the first necessary condition.

2. Similarly, if $e_2(\mathbf{x})$ is the largest excess, then it reaches the minimum when $e_2(\mathbf{x}) = e_{13}(\mathbf{x})$ or $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. We can also analogously show that the third necessary condition including (7) and (8) corresponds to the case that $e_2(\mathbf{x}) = e_{13}(\mathbf{x})$.
3. Similarly, if $e_3(\mathbf{x})$ is the largest excess, then it arrives to the minimum when $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$ or $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. We can also show that the second necessary condition including (5) and (6) corresponds to the case that $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$.
4. If $e_{12}(\mathbf{x})$ is the largest excess, then according to (2) we can decrease the value of x_3 to reduce this excess. However, from (2) we find that decreasing x_3 shall raise the excess $e_3(\mathbf{x})$. Furthermore, since $x_1 + x_2 + x_3 = v(123)$, we should increase x_1 and x_2, so increasing $e_{23}(\mathbf{x})$ and $e_{13}(\mathbf{x})$ in (2). Note that the excesses $e_1(\mathbf{x})$ and/or $e_2(\mathbf{x})$ in (1) decrease when we increase x_2 and/or x_3 to reduce the largest excess $e_{12}(\mathbf{x})$. Thus, the largest excess reaches the minimum when $e_{12}(\mathbf{x}) = e_3(\mathbf{x})$ or $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$.

   We have shown that the second necessary condition corresponds to the case that $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$. Next we use (2) to solve $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$, and obtain (11). Since $e_{12}(\mathbf{x})$ is the largest excess, we have $e_{12}(\mathbf{x}) - e_i(\mathbf{x}) \geq 0$, for $i = 1, 2, 3$; and we use (11) to simplify these three inequalities and reach (12). Hence, the fifth necessary condition including (11) and (12) corresponds to the case that $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$.
5. Similarly, if $e_{13}(\mathbf{x})$ is the largest excess, then it reaches the minimum when $e_{13}(\mathbf{x}) = e_2(\mathbf{x})$ or $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$; the former corresponds to the third necessary condition and the latter corresponds to the fifth necessary condition.
6. Similarly, if $e_{23}(\mathbf{x})$ is the largest excess, then it reaches the minimum when $e_{23}(\mathbf{x}) = e_1(\mathbf{x})$ or $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$; the former corresponds to the fourth necessary condition and the latter corresponds to the fifth necessary condition.

This proves the lemma. ■

Appendix E Proof of Theorem 2

We can easily find from Lemma 1 that if $v(123) \geq \max(3v(12),\ 3v(13),\ 3v(23))$, the excesses $e_i(\mathbf{x})$ (for $i = 1,\ 2,\ 3$) are the largest and thus the imputation $\mathbf{x} = (x_1, x_2, x_3) = (v(123)/3, v(123)/3, v(123)/3)$ when the largest excesses are reduced to the minimum. Since we have obtained the values of x_i, for $i = 1,\ 2,\ 3$, we cannot decrease any other excess. Hence, we arrive to Case 1 in Theorem 2.

Next, we consider the situation in which the largest excess is minimized because the second condition in Lemma 1 is satisfied. Under the condition, $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$, $x_3 = [v(123) - v(12)] \diagup 2$ and the value of x_2 is determined under the constraint (6). By using (6), we consider the following four cases in which we minimize the second largest excesses.

1. If $v(123) \geq v(12) + 2v(23)$, $v(123) \geq v(12) + 2v(13)$ and $v(123) \leq 3v(12)$, then $\{[v(123) + v(12)]/2 - v(13)\} \geq v(12) \geq [v(123) - v(12)]/2 \geq v(23)$, and we can reduce (6) to $[v(123) - v(12)]/2 \leq x_2 \leq v(12)$ and we can easily show that

$$\max\{v(23), v(13)\} \leq [v(123) - v(12)]/2 \leq v(12). \tag{13}$$

Next, we choose an appropriate value of x_2 to minimize the second largest excesses subject to $[v(123) - v(12)]/2 \leq x_2 \leq v(12)$. Except for the largest excesses $e_3(\mathbf{x})$ and $e_{12}(\mathbf{x})$, the other excesses are computed as

$$\begin{aligned}
e_1(\mathbf{x}) &= -x_1 = x_2 - \frac{v(123) + v(12)}{2}, && (14)\\
e_2(\mathbf{x}) &= -x_2, && (15)\\
e_{13}(\mathbf{x}) &= x_2 - v(123) + v(13),\\
e_{23}(\mathbf{x}) &= v(23) - v(123) + x_1 = v(23) - \frac{v(123) - v(12)}{2} - x_2.
\end{aligned}$$

Using (13) we have $e_1(\mathbf{x}) \geq e_{13}(\mathbf{x})$ and $e_2(\mathbf{x}) \geq e_{23}(\mathbf{x})$, which implies that $e_1(\mathbf{x})$ and/or $e_2(\mathbf{x})$ could be the second largest excess. From (14) and (15) we find that the second largest excesses are reduced to the minimum as $e_1(\mathbf{x}) = e_2(\mathbf{x})$, or, $x_2 = [v(123) + v(12)]/4$, which satisfies the constraint $[v(123) - v(12)]/2 \leq x_2 \leq v(12)$. Since $x_1 + x_2 + x_3 = v(123)$, we compute $x_1 = x_2 = [v(123) + v(12)]/4$. We notice that the other excesses (i.e., $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$) cannot be reduced because the imputation $\mathbf{x}$ has been determined; thus, the nucleolus solution is $y_1 = y_2 = [v(123) + v(12)]/4$ and $y_3 = [v(123) - v(12)]/2$, which corresponds to the second case (with $i, j = 1, 2$ and $i \neq j$, and $k = 3$) in Theorem 2.

2. If $v(123) \geq v(12)+2v(23)$, $v(123) \leq v(12)+2v(13)$ and $v(12) \geq v(13)$, then $v(12) \geq \{[v(123)+v(12)]/2-v(13)\} \geq [v(123)-v(12)]/2 \geq v(23)$, and we can reduce (6) to $[v(123)-v(12)]/2 \leq x_2 \leq \{[v(123) + v(12)]/2 - v(13)\}$. Similar to the last case, we can show that under this

condition the nucleolus solution is computed as $y = (y_1, y_2, y_3) = ([v(12)+v(13)]/2, [v(123) - v(13)]/2, [v(123) - v(12)]/2)$, which corresponds to the third case (with $i = 1$, $j = 2$ and $k = 3$) in Theorem 2.

3. If $v(123) \leq v(12) + 2v(23)$, $v(123) \geq v(12) + 2v(13)$ and $v(12) \geq v(23)$, then we find the formula of computing nucleolus solution for the third case (with $i = 2$, $j = 1$ and $k = 3$) in Theorem 2.
4. If $v(123) \leq v(12) + 2v(23)$, $v(123) \leq v(12) + 2v(13)$ and $v(123) + v(12) \geq 2[v(13) + v(23)]$, then we find the formula of computing nucleolus solution for the fourth case (with $i, j = 1, 2$ and $i \neq j$, and $k = 3$) in Theorem 2.

Similar to our above analysis, we can analyze the third and fourth conditions in Lemma 1, and reach the corresponding results in Theorem 2.

From Lemma 1 we find that under the condition (12), the excesses $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ are the largest and the triple imputation x is obtained as (11). Thus, we cannot decrease any other excess. Hence, we arrive to fifth case in Theorem 2. ■