Game-Theoretic Analyses of Decentralized Assembly Supply Chains: Non-Cooperative Equilibria vs. Coordination with Cost-Sharing Contracts[1]

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Abstract

This paper considers a multiple-supplier, single manufacturer assembly supply chain where the suppliers produce components of a short life-cycle product which is assembled by the manufacturer. In this single-period problem the suppliers determine their production quantities and the manufacturer chooses the retail price. We assume that the manufacturer faces a random price-dependent demand in either additive or multiplicative form. For each case, we analyze both simultaneous-move and leader-follower games to respectively determine the Nash and Stackelberg equilibria, and find the globally optimal solution that maximizes the system-wide expected profit. Then, we introduce appropriate buy-back and lost sales cost-sharing contracts to coordinate this assembly supply chain, so that when all the suppliers and the manufacturer adopt their equilibrium solutions, the system-wide expected profit is maximized.

Key words: Assembly supply chain, game theory, buy-back, lost-sales cost-sharing.

1 Introduction

In many supply chains assembled products are composed of complementary components. It is well-known that a large number of firms in industry outsource the production of the components to external suppliers in order to reduce costs and increase production flexibility. For example, in the U.S., Toyota outsources the production of car components to many suppliers who then deliver the components to Toyota's assembly plant in Kentucky (Chopra and Meindl [7]). In particular, more than 75% of the parts and 98% of the steel used in the production of vehicles at this assembly plant come from U.S. suppliers. In 2005, the plant had 350 suppliers across the continental United States. The Toyota production system (one of the first successful examples of a Just-In-Time system) is "all about producing only what's needed and transferring only what's needed." By adopting such an efficient system known as a "pull"-type supply chain, Toyota's assembly plant uses the components (delivered by its suppliers) to assemble final products when the orders of its customers (e.g., dealers) arrive; see, for example, Chopra and Meindl [7] and Simchi-Levi et al. [17]. As reported by Reinhardt in [15], Nokia also recently implemented the pull strategy to make built-to-order phones each with a unique faceplate with the operator's logo on it and special keypad buttons that take users directly to certain wireless services, etc. Another well-known example is Dell which adopts the pull-type strategy to assemble computers only when its customers' orders arrive online.

As Cachon [4] and Granot and Yin [8] have discussed, in a pull-type assembly supply chain all suppliers of the assembly plant need to determine their production quantities of components while the assembly plant needs to choose its sale price of final product. Moreover, in order to quickly respond to its customers, the plant also aims to strengthen partnering relationship with its suppliers and establish coordination with the suppliers for system-wide improvement. As another real example of a manufacturer's effort to induce supply chain coordination, in 2006, Motorola decided to spend $60 million in Singapore to centralize and streamline global supply chain operations with its suppliers and customers. As outsourcing is considered to be one of the strategies achieving supply chain integration, there is an extensive literature focusing on outsourcing strategies and vertical integration in supply chains; see, Cachon and Harker [5].

As the above examples illustrate, the coordination of a decentralized supply chain with assembled products appears to be an interesting and important problem worth investigating. Motivated by these examples, in this paper we consider the following natural question: What mechanism can be developed to coordinate all members in such a supply chain? As Cachon [3] indicated, supply chain coordination is achieved if and only if all firms in a decentralized supply chain can behave (that is, make decisions) as if they are operating in a centralized supply chain. More specifically, in a decentralized supply chain, all firms primarily aim at optimizing their own individual objectives rather than the chainwide objective, thus their self-serving focus may result in a deterioration of the chainwide performance. To improve the supply chain's performance, a proper mechanism must be developed to coordinate all channel members so that both the individual supply chain members' objectives and the chainwide performance can be optimized; see, e.g., Leng and Zhu [10]. A common (and useful) mechanism for supply chain coordination is to develop a set of properly-designed contracts among all supply chain members (Cachon [3]). With the successful use of this mechanism the last decade has witnessed a rapidly increasing interest in supply chain coordination with contracts.

In this paper, we restrict our attention to a multiple-supplier, one-manufacturer supply chain with complementary products. (Hereafter, such a supply chain with complementary products is called an *assembly* supply chain, as in Carr and Karmarkar [6].) In this assembly supply chain, multiple suppliers produce their complementary components, and serve a common manufacturer who assembles the final products with short life cycles and satisfies a random demand. Our assumption of short product life cycles reflects the following fact: The last two decades have witnessed rapid technological innovation and a high level of competition in the marketplace. In response to these developments, many firms (e.g., manufacturers of personal computers, cell phones, cars, etc.) have implemented the philosophy of "life-cycle management" to reduce life cycles of their products. As in many publications (such as Linh and Hong [11] and Parlar and Weng [12]) concerned with the assembly supply chains with short product life cycles, we construct our model and perform our analysis using the single-period (newsboy) setting. Similar to the *pull*-type system discussed in Cachon [4] and Granot and Yin [8], the $n(\geq 2)$ suppliers determine their production quantities independently of each other. Moreover, we assume that all members of this supply chain are risk-neutral and the demand is only sensitive to the retail price chosen by the manufacturer. Note that, in practice, consumers' demands may also depend on some other factors (e.g., the quality of the product that the consumers buy). However, in this paper, we only focus on the manufacturer's pricing decision and the suppliers' quantity decisions, as in most of previous publications regarding assembly supply chains. Accordingly, we assume that the demand is only dependent of the manufacturer's retail price, and thus use Petruzzi and Dada's additive and multiplicative demand forms [14]—which have been commonly used to analyze assembly supply chains with price-dependent demand—to characterize the random demand.

We use *buy-back* and *lost-sales cost-sharing* contracts between the n suppliers and the manufacturer to coordinate the supply chain. With the buy-back contract, the manufacturer returns the unused components to the suppliers at the buy-back price. Because all unused components can be returned to the suppliers (albeit at some loss), the manufacturer does not concern himself with the optimal order quantities of the components. Instead, he attempts to choose the optimal price to maximize his expected profit. Buy-back contracts have been widely used to analyze supply chain coordination. As described in Cachon [3], a typical buy-back contract (also called *return policies*) has two parameters; the ith supplier's wholesale price, w_i, and the buy-back price, v_i, $i = 1, \dots, n$. Under such a contract, supplier i charges the manufacturer w_i per unit purchased at the beginning of the single period, and pays the manufacturer v_i per unit remaining at the end of the period. In our paper, we model the buy-back contract as a vector $(\mathbf{w}, \mathbf{v})$, where $\mathbf{w} = (w_1, \dots, w_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. For an early application of buy-back contracts, see Pasternack [13].

With the lost-sales cost-sharing contract, when shortages arise, suppliers and the manufacturer share the shortage cost. Assuming that one unit of the final product needs one unit of each of the n components, the manufacturer needs equal number of components for assembly from each supplier. If the number of components received from the suppliers happens to be different, the number of the final product the manufacturer can assemble equals the minimum of these quantities. In our lost-sales cost-sharing contract, the shortage penalty cost incurred by the manufacturer at the end of the single period is shared among all members of this supply chain. Given a unit shortage (underage) cost u, we define the percentage of this cost absorbed by the ith supplier as $\phi_i \in [0,1]$ with $\phi = \sum_{i=1}^{n} \phi_i \in [0,1]$. Thus, all suppliers pay the shortage cost ϕu, and the manufacturer bears the cost of $(1-\phi)u$. The lost-sales cost-sharing contract is characterized by the vector $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$. See Table 1 for a complete list of the notation used in this paper.

Symbol	**Description**
c_i	Unit production cost of supplier $i = 1, \ldots, n$.
$D(p, \varepsilon)$	Price-dependent random demand in the single period. For the additive case, $D(p,\varepsilon) = y(p) + \varepsilon$; for the multiplicative case $D(p,\varepsilon) = y(p)\varepsilon$.
$y(p)$	Deterministic component of the random demand. For the additive case, $y(p) = a - bp$ (a, $b > 0$); for the multiplicative case, $y(p) = ap^{-b}$ ($a > 0$, $b > 1$).
ε	Error term with c.d.f. $F(\cdot)$ and p.d.f. $f(\cdot)$ taking values in the range $[A, B]$ with $A > -a$ for the additive case and with $A > 0$ for the multiplicative case. The mean value and variance of ε are denoted by μ and σ^2, respectively, i.e., $E(\varepsilon) = \mu$ and $\mathrm{Var}(\varepsilon) = \sigma^2$.
ϕ_i	Percentage of underage cost u absorbed by supplier $i = 1, \ldots, n$, (*contract parameter*).
$\phi = \sum_{i=1}^{n} \phi_i$	Fraction of underage cost absorbed by all suppliers, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$.
m	Manufacturer's assembly cost per unit.
p	Retail price of the manufacturer.
Π	System-wide profit.
Π_M	Manufacturer's random profit.
Π_{S_i}	Supplier i's random profit $i = 1, \ldots, n$.
q_i	Production quantity of supplier $i = 1, \ldots, n$.
q	Production quantity of all suppliers when they produce the same amount.
$Q = \min_{i=1,\ldots,n}(q_i)$	Number of each component received by manufacturer.
s_i	Salvage value for supplier $i = 1, \ldots, n$ for unsold components.
S_i	Supplier $i = 1, \ldots, n$.
u	Underage cost per unit for lost sales.
v_i	Supplier i's buy-back price, $i = 1, \ldots, n$ (*contract parameter*).
$v = \sum_{i=1}^{n} v_i$	Total unit buy-back price paid by n suppliers, $\mathbf{v} = (v_1, \ldots, v_n)$.
w_i	Supplier i's wholesale price, $i = 1, \ldots, n$ (*contract parameter*).
$w = \sum_{i=1}^{n} w_i$	Manufacturer's purchase cost for all the components that make up one unit of the final product, $\mathbf{w} = (w_1, \ldots, w_n)$.

Table 1: List of notations.

Our paper uses game theory to analyze non-cooperative and coordinated assembly supply chains. In practical applications the manufacturer and n suppliers may simultaneously or sequentially make optimal decisions to maximize their individual expected profits. Accordingly, we analyze both a simultaneous-move game (in which all members of the assembly supply chain make their decisions concurrently) and a leader-follower game (in which the manufacturer announces its pricing decision before n suppliers make their production decisions). We use the Nash and Stackelberg equilibria to characterize all supply chain members' optimal decisions for the simultaneous-move and leader-follower games, respectively. Note that Wang [18] also considered the two games but only with the multiplicative demand form, and used different contracts to induce supply chain coordination.

Since the supply chain members make optimal decisions to maximize their individual profits, the system-wide expected profit is usually lower than the case when they coordinate their decisions. Thus, to improve the supply chain's performance, we design buy-back and lost-sales cost-sharing contracts which achieve supply chain coordination. Under properly designed contracts, all supply chain members choose their equilibrium solutions and the maximum system-wide profit is realized.

In a recent literature review [9], Leng and Parlar surveyed a large number of publications that focus on supply chain-related game problems with substitutable products and indicate that there are only a few papers concerned with game-theoretic models for complementary products. We now briefly review some of the important papers that used game-theoretic models in assembly supply chains. Wang [18] considered joint pricing-production decision problems in supply chains with complementary products for a single period. Wang adopted a multiplicative demand model which is sensitive to sale price, and incorporated the consignment-sales and revenue-sharing contracts into both simultaneous-move and leader-follower games. Since Wang [18] analyzed assembly supply chains with the multiplicative demand form, in this paper we focus our analysis on the additive demand case, and provide only the major results for the multiplicative demand case without specific discussion.

Wang and Gerchak [19] examined a pricing-capacity decision problem for an assembly supply chain with an assembler and multiple suppliers. Assuming that the demand is random but it is independent of price, the authors considered two game settings: The first is one where the assembler sets the prices, and the second is for the suppliers to simultaneously select the prices each wants to charge for its component. Bernstein and DeCroix [1] considered an assembly supply chain where two components are used to assemble a single final product that is then sold by an assembler to meet the random, price-independent demand. The authors investigated the equilibrium base-stock levels for the assembler and two component suppliers, and described a payment scheme to coordinate the assembly supply chain. Granot and Yin [8] investigated competition and cooperation in a multiple-supplier, one-manufacturer supply chain with complementary products. For the pull and push systems, these authors considered two levels of problems: at the first level, they used the concepts of Nash equilibrium and farsighted stability to identify stable coalitional structures among suppliers; and at the second level they developed a Stackelberg game to examine the interactions between the manufacturer and suppliers.

The remainder of this paper is organized as follows: In Section 2, we assume that the random price-dependent demand is given in additive form, and design a set of buyback and lost-sales cost-sharing contracts to coordinate the assembly supply chain for both simultaneous-move and leader-follower games. In Section 3, we assume that the price-dependent demand is also random but it assumes the multiplicative form. For this case, we show that the supply chain can always be coordinated by a set of properly-designed contracts for both simultaneous-move and leader-follower games. The paper ends with conclusions in Section 4.

2 Non-Cooperative Equilibria and Supply Chain Coordination with Price-Dependent Random Demand: Additive Form

In this section, the manufacturer assembles the suppliers' components to satisfy random price-dependent demand in an additive form. We consider the following two games: (i) a "simultaneous-move" game where the manufacturer and the suppliers concurrently make their decisions without any communication; (ii) a "leader-follower" game in which the manufacturer, as the leader, first announces his pricing decision and the suppliers, as the followers, respond to the leader's decision. For the former game, we compute the Nash equilibrium; and for the latter, we compute the Stackelberg equilibrium for the supply chain. To coordinate this assembly supply chain, we then find the globally-optimal solution and develop a properly-designed buyback and lost-sales cost-sharing contract that can achieve supply chain coordinate.

2.1 Suppliers' Production and Manufacturer's Pricing Decisions with No Coordination: Nash vs. Stackelberg Equilibria

We now consider a non-cooperative case where n suppliers (denoted by S_i, $i = 1, \ldots, n$) and the manufacturer determine their respective production quantities q_i and the retail price p independently without supply chain coordination.

Using the pull-type strategy, the manufacturer receives $Q = \min(q_1, q_2, \ldots, q_n)$ units from each supplier (because the assembly of a final product only needs one unit of each component), and assembles $\min(Q, D)$ units of final products (where D denotes the realized demand during the single period). If D is more than Q, then the manufacturer assembles Q units of final products, and incurs a unit underage cost u for each unit of unsatisfied demand. According to the lost-sales cost-sharing contract that is in place, the percentage of this cost absorbed by S_i $(i = 1, 2, \ldots, n)$ is $\phi_i \in [0, 1]$, and the manufacturer's percentage is $1 - \phi = 1 - \sum_{i=1}^{n} \phi_i$. However, if Q is more than D, then, according to the buyback contract, supplier S_i buys back the unsold components at the unit price of v_i, $i = 1, \ldots, n$. Since S_i produces the component i at the unit production cost c_i and sells them to the manufacturer at the unit wholesale price w_i, S_i receives the net profit of $w_i - c_i > 0$ for each unit component sold to the manufacturer. Here, we assume that $v_i \geq w_i - c_i$, and thus $v_i - (w_i - c_i)$ is the penalty cost incurred by S_i when his production quantity q_i exceeds D by one unit. We make this assumption to assure that a buy back agreement should not benefit a supplier.

In this section we assume that the demand is random in an additive form and depends on price p. As in Petruzzi and Dada [14], the price-sensitive random demand in a commonly-used additive form in the newsvendor context is formulated as

$$D(p, \varepsilon) = y(p) + \varepsilon, \tag{1}$$

where the deterministic term $y(p)$ and the error term ε are defined in Table 1. Note that, as indicated in Table 1, we assume that the error term has a lower bound A and an upper bound B. This assumption is reasonable because, in reality, the market size cannot be infinite and thus the upper bound B is taken as finite. Moreover, in order to assure that $D(p, \varepsilon)$ is positive, the lower bound A must be greater than $-a$ for the additive demand case. The assumption has been widely used in the literature; see, for example, Petruzzi and Dada [14]. Since $A > -a$, it follows that $D(p, \varepsilon) \geq (A + a) - bp$ and thus to ensure that demand always assumes non-negative values (even though $y(p)$ could be negative when $p \geq a/b$), the manufacturer's retail price p should be smaller than or equal to $(a + A)/b$, i.e., $p \leq (a + A)/b$. In fact, when the manufacturer maximizes its profit, the optimal price should always satisfy the inequality because the manufacturer's profit with a non-positive demand cannot be maximum.

The demand function (1) is "common knowledge" for all suppliers and the manufacturer. Thus, all suppliers know the demand range $[y(p) + A, y(p) + B]$, and determine their quantities subject to this range, i.e., $y(p) + A \leq q_i \leq y(p) + B$, $i = 1, \ldots, n$.

2.1.1 Suppliers' Best Response Production Decisions

The supplier S_i produces q_i units of component $i = 1, \ldots, n$ independently from others at a total production cost of $c_i q_i$ to maximize his/her expected profit. Since the manufacturer receives the minimum quantity $\min_{j=1,...,n}\{q_j\}$ at a wholesale price of w_i from supplier S_i, the sale revenue of S_i is $w_i Q = w_i \min_{j=1,...,n}\{q_j\}$.

When supplier S_i's production quantity is not the minimum, this supplier disposes the remaining components at the unit salvage value s_i with $s_i < c_i$. As discussed in Section 1, if the demand for the final product falls short of the available units $Q = \min_{j=1,...,n}\{q_j\}$, then supplier S_i buys back his unsold components at the price $v_i \in [w_i - c_i, w_i]$. Thus, S_i's buyback cost is $v_i[Q - D(p, \varepsilon)]^+$. On the other hand, if the realized demand is more than the manufacturer's available quantity Q, then S_i pays the unit cost of $\phi_i u$ for lost sales. For this case, the lost sales cost incurred by S_i is $\phi_i u[D(p, \varepsilon) - Q]^+$.

Thus, supplier S_i's net random profit Π_{S_i} is obtained as sale revenue plus the salvage value, minus the production cost, buyback costs and partial lost sales cost as

$$\Pi_{S_i} = w_i Q + s_i(q_i - Q_{-i})^+ - c_i q_i - v_i[Q - D(p, \varepsilon)]^+ - \phi_i u[D(p, \varepsilon) - Q]^+, \tag{2}$$

where $Q_{-i} \equiv \min\{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n\}$.

Theorem 1 For given contract parameters $(\mathbf{v}, \boldsymbol{\phi})$, the manufacturer's pricing decision p, all suppliers' best production quantities that maximize their expected profits are equal, i.e.,

$$q_1^B = \cdots = q_n^B = y(p) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right), \tag{3}$$

where

$$\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u} = \min_{i=1,\ldots,n}\left(\frac{w_i - c_i + \phi_i u}{v_i + \phi_i u}\right), \tag{4}$$

and the index k corresponds to that supplier with the minimum value for the right-hand-side term in (4).

Proof. For a proof of this theorem and the proofs of all subsequent theorems, see online Appendix A. ■

2.1.2 Manufacturer's Best Response Pricing Decision

The manufacturer receives $Q = \min(q_1, q_2, \ldots, q_n)$ units from each supplier and determines the unit retail price p. Using the pull-type strategy, the manufacturer incurs the assembly cost m for each unit of satisfied demand, and returns the unused components to suppliers at the end of the single period. The manufacturer's net profit is computed as the revenue from sales, plus the buyback income, minus the procurement, assembly and the lost-sales cost incurred by the manufacturer. Thus, the manufacturer's random profit function, Π_M, is given as

$$\begin{aligned}
\Pi_M &= p\min[Q, D(p,\varepsilon)] + v[Q - D(p,\varepsilon)]^+ - wQ \\
&\quad -m\min[Q, D(p,\varepsilon)] - (1-\phi)u[D(p,\varepsilon) - Q]^+ \\
&= (p-m)\min[Q, D(p,\varepsilon)] + v[Q - D(p,\varepsilon)]^+ - wQ \\
&\quad -(1-\phi)u[D(p,\varepsilon) - Q]^+, \qquad (5)
\end{aligned}$$

where $w = \sum_{i=1}^n w_i$ is the manufacturer's purchase cost for all the components that make up one unit of the final product, and $v = \sum_{i=1}^n v_i$ is the total unit buy-back price paid by n suppliers.

Expected profit of the manufacturer can now be found as follows:

$$\begin{aligned}
E(\Pi_M) &= (p-m)E\{\min[Q, D(p,\varepsilon)]\} + vE\{[Q - D(p,\varepsilon)]^+\} - wQ \\
&\quad -(1-\phi)uE\{[D(p,\varepsilon) - Q]^+\} \\
&= (p-m-v)[y(p)+\mu] + (v-w)Q + [p-m+(1-\phi)u - v] \\
&\quad \times \int_{Q-y(p)}^{B} [Q - y(p) - x]f(x)dx. \qquad (6)
\end{aligned}$$

Next, for both simultaneous-move and leader-follower games, we analyze the manufacturer's best response to the suppliers' equal production quantities. As we argued before, the manufacturer's order quantity is the minimum of the suppliers' production quantities. To simplify the analysis, we define

$$z_i \equiv \frac{w_i - c_i + \phi_i u}{v_i + \phi_i u}, \quad i = 1, \ldots, n,$$

so that $Q = y(p) + F^{-1}(z_k)$, where, as before, the index k corresponds to that supplier with the minimum value for the right-hand-side term in (4), i.e., $z_k = \min_{i=1,\ldots,n} z_i$.

Theorem 2 For the given contract parameters $(\mathbf{w}, \mathbf{v}; \boldsymbol{\phi})$ and suppliers' quantity decisions $q = (q_1, \ldots, q_n)$, we find the manufacturer's best response price p^B as follows:

1. For the simultaneous-move game, the manufacturer's best response price p^B is determined as the *unique* solution of the nonlinear equation

$$\int_A^{Q-y(p)} F(x)dx + [p-m+(1-\phi)u - v]bF(Q-y(p)) = Q + (1-\phi)ub. \qquad (7)$$

2. For the leader-follower game, the manufacturer's best response price p^B is determined as

the *unique* solution of the nonlinear equation

$$\int_A^{Q-y(p)} F(x)dx + (p-m-w)b = Q. \quad \blacktriangleleft \tag{8}$$

2.1.3 Nash and Stackelberg Equilibria

In the analysis above, we have derived each supplier's and the manufacturer's best responses in Theorems 1 and 2, given the other players' decisions. Using the best response functions we now find the Nash equilibrium for the simultaneous game and the Stackelberg equilibrium for the leader-follower game.

Theorem 3 For the additive demand case, we find the Nash and Stackelberg equilibria as follows:

1. For the simultaneous-move game, the Nash equilibrium $(p^N, q_1^N, \ldots, q_n^N)$ is found as

$$p^N = m-(1-\phi)u+v+\frac{1}{bz_k}\left[q^N+(1-\phi)ub-\int_A^{F^{-1}(z_k)} F(x)dx\right], \tag{9}$$

$$q_i^N = \frac{[a+b(1-\phi)u-bm-bv]z_k-b(1-\phi)u}{z_k+1} + \frac{\int_A^{F^{-1}(z_k)} F(x)dx + z_k F^{-1}(z_k)}{z_k+1}, \tag{10}$$

   for $i=1,\ldots,n$ and $q^N \equiv q_1^N = \cdots = q_n^N$.

2. For the leader-follower game, the Stackelberg equilibrium $(p^S, q_1^S, \ldots, q_n^S)$ is found as

$$p^S = \frac{1}{2b}\left[a+b(m+w)+F^{-1}(z_k)-\int_A^{F^{-1}(z_k)} F(x)dx\right], \tag{11}$$

$$q_i^S = \frac{1}{2}\left[a-b(m+w)+F^{-1}(z_k)+\int_A^{F^{-1}(z_k)} F(x)dx\right], \tag{12}$$

   for $i=1,\ldots,n$ and $q^S \equiv q_1^S = \cdots = q_n^S$. $\blacktriangleleft$

2.2 Supply Chain Coordination with Buy-Back and Lost-Sales Cost-Sharing Contracts

It is important to note that the Nash and Stackelberg equilibria (found in Theorem 3) may not result in supply chain coordination under which the system-wide profit is maximized. To coordinate this supply chain, we now design buy-back and lost-sales cost-sharing contracts so that the resulting Nash and Stackelberg equilibria shall be both identical to the globally optimal solution. In particular, in this section we find the properly designed values of $\mathbf{w}$ (suppliers' wholesale prices), $\mathbf{v}$ (suppliers' buy-back prices), and $\boldsymbol{\phi}$ (lost-sales cost sharing parameters) which will make the decentralized assembly supply chain behave like a centralized one.

2.2.1 Total Profit for the Assembly Supply Chain

We first find the optimal solution that maximizes the total profit for the assembly supply chain *as if* the supply chain members were acting in a centralized fashion. For such a system we denote the globally optimal retail price by p^* and the production quantity of each supplier by q_i^*, $i = 1, 2, \ldots, n$. The system-wide profit function is then the sum of (5) and (2), i.e.,

$$\begin{aligned}\Pi &= \Pi_M + \sum_{i=1}^{n} \Pi_{S_i} \\ &= (p-m)\min[Q, D(p,\varepsilon)] - u[D(p,\varepsilon) - Q]^+ - \sum_{i=1}^{n}[c_i q_i - s_i(q_i - Q_{-i})^+]. \qquad (13)\end{aligned}$$

The expected profit function is thus

$$E(\Pi) = (p-m)E\{\min[Q, D(p,\varepsilon)]\} - uE[D(p,\varepsilon) - Q]^+ - \sum_{i=1}^{n}[c_i q_i - s_i(q_i - Q_{-i})^+].$$

To maximize the total expected profit, all suppliers' production quantities should be equal, i.e., $q_1 = \cdots = q_n$; otherwise, some suppliers would incur penalty costs for the leftovers. Letting $q = q_i$, $i = 1, \ldots, n$, denote the production quantity of each supplier, we reduce the expected system-wide profit function to

$$\begin{aligned}E(\Pi) &= (p-m)E\{\min[q, D(p,\varepsilon)]\} - uE[D(p,\varepsilon) - q]^+ - cq \\ &= (p-m)[y(p) + \mu] - cq + (p-m+u)\int_{q-y(p)}^{B}[q - y(p) - x]f(x)dx\end{aligned}$$

where $c = \sum_{i=1}^{n} c_i$.

Theorem 4 In the assembly supply chain, the globally-optimal retail price p^* is determined by solving the following equation

$$2bp^* + \int_{A}^{F^{-1}(\xi(p^*))} F(x)dx - F^{-1}(\xi(p^*)) = a + b(m+c),$$

and the optimal production quantity q^* is found as

$$q^* = y(p^*) + F^{-1}(\xi(p^*))$$

where

$$\xi(p^*) = \frac{p^* - m + u - c}{p^* - m + u}. \quad \blacktriangleleft \qquad (14)$$

The following corollary gives an upper bound for the optimal price p^* in terms of the contract design parameters.

Corollary 1 The optimal retail price p^* is bounded from above by $[a + b(m+c) + B]/(2b)$.

Proof. From Theorem 4, we have

$$\begin{aligned} p^* &= \left[a + b(m+c) + F^{-1}(\xi(p)) - \int_A^{F^{-1}(\xi(p))} F(x)dx\right] \frac{1}{2b}. \\ &< [a + b(m+c) + B] \frac{1}{2b}. \end{aligned}$$

This follows since $F^{-1}(\xi(p)) \le B$ and $\int_A^{F^{-1}(\xi(p))} F(x)dx > 0$. ■

2.2.2 Design of Buy-Back and Lost-Sales Cost-Sharing Contracts for the Simultaneous-Move Game

We now consider the simultaneous-move game, and examine the impact of the buy-back and lost-sales cost-sharing contracts on supply chain coordination. In particular, we investigate whether or not this multiple-supplier, one-manufacturer assembly supply chain can be coordinated under a pair of the properly-designed contracts. With the appropriate values of contract parameters $(\mathbf{w}, \mathbf{v}; \boldsymbol{\phi})$, each member of the supply chain chooses his/her equilibrium solution, which is identical to the globally optimal solution that maximizes the system-wide expected profit found in Theorem 4. If the proper contracts exist, we attempt to find them.

Prior to commencing the design of the contract, we present a discussion of the unit underage cost u. As usual, the value of u is estimated as the sum of the opportunity cost of each unsatisfied demand and the goodwill cost per unit unsatisfied demand. The total cost of assembling one unit of the final product is $m + c$ and p is the unit retail price, thus $u = p - (m + c) + g$, where g is the goodwill cost. Since the goodwill cost is $g \ge 0$ (but difficult to measure), we impose the condition that the underage cost should have the property $u \ge p - (m + c)$.

However, Theorem 4 shows that the globally optimal price is determined in terms of a given value of u. Since the optimal value of p is not known *a priori*, the requirement $u \ge p - (m + c)$ may not be always satisfied. But using the result of Corollary 1, we see that choosing an underage cost with the property $u \ge [a + b(m + c) + B]/2b - (m + c)$ will always satisfy the requirement $u \ge p - (m + c)$.

Now, in order to coordinate the supply chain, the Nash equilibrium must be equal to the globally optimal solution, i.e., $p^N = p^*$ and $q^N = q^*$. From Theorem 3, we know that the Nash equilibrium (p^N, q^N) is obtained by solving

$$\begin{aligned} & \int_A^{q^N - y(p^N)} F(x)dx + [p^N - m + (1-\phi)u - v]bF(q^N - y(p^N)) \\ &= q^N + (1-\phi)ub, \qquad (15) \\ q^N &= y(p^N) + F^{-1}(z_k), \qquad (16) \end{aligned}$$

where, as before, $z_k = \min_{i=1,\ldots,n} z_i$. Similarly, from Theorem 4, the globally optimal solution

(p^*, q^*) is found by solving

$$\begin{aligned} & \int_A^{q^*-y(p^*)} F(x)dx + b(p^* - m + u)F(q^* - y(p^*)) \\ = & \; q^* + bu, \qquad (17) \\ q^* = & \; y(p^*) + F^{-1}\left(\frac{p^* - m + u - c}{p^* - m + u}\right). \qquad (18) \end{aligned}$$

Hence, if and only if the two sets of equations above are the same, i.e., if (15) is identical to (17) and if (16) is identical to (18), the Nash equilibrium will be identical to the globally optimal solution. Equating the stated equations and simplifying, the conditions that will assure $p^N = p^*$, and $q^N = q^*$ are found as

$$vz_k = (1 - z_k)\phi u \quad \text{and} \quad \frac{p^* - m + u - c}{p^* - m + u} = \frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}.$$

Defining $z \equiv (p^* - m + u - c)/(p^* - m + u)$, we find $z = z_k$ and we re-write the above conditions as:

$$zv = (1 - z)\phi u \quad \text{and} \quad z = \frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}. \qquad (19)$$

Furthermore, the chosen values of parameters which satisfy the above conditions should also meet the following five requirements:

$$\begin{cases} (1): & w_i - c_i \le v_i \le w_i, \quad i = 1, \ldots, n. \\ (2): & 0 \le \phi_i \le 1, \quad i = 1, \ldots, n. \\ (3): & \sum_{i=1}^n \phi_i \le 1. \\ (4): & \dfrac{w_i - c_i + \phi_i u}{v_i + \phi_i u} \ge z, \quad i = 1, \ldots, k-1, k+1, \ldots, n, \\ (5): & w_i \ge c_i, \quad i = 1, \ldots, n. \end{cases} \qquad (20)$$

Theorem 5 For the simultaneous-move game with the additive demand form **(1),** supply chain coordination can be achieved by a pair of the properly-designed buy-back and lost-sales cost-sharing contract with the following parameter values:

- The values of S_k's parameters are given as

$$w_k = (1 + z)c_k, \quad v_k = c_k, \quad \text{and} \quad \phi_k = 0, \qquad (21)$$

  where S_k is chosen as the supplier with the smallest unit production cost, i.e., $c_k = \min_{i=1,\ldots,n} c_i$.
- The values of S_i's parameters ($i = 1, \ldots, k-1, k+1, \ldots, n$) are given as

$$\begin{cases} w_i = v_i + c_i = \left(\dfrac{1-z}{cz}u + 1\right)c_i + \left(\dfrac{1-z}{cz}u - 1\right)\dfrac{c_k}{n-1}, \\ v_i = \dfrac{1-z}{cz}uc_i + \left(\dfrac{1-z}{cz}u - 1\right)\dfrac{c_k}{n-1}, \\ \phi_i = \dfrac{1}{c}\left(c_i + \dfrac{c_k}{n-1}\right). \quad \blacktriangleleft \end{cases} \qquad (22)$$

Remark 1 Theorem 5 indicates that, for the simultaneous-move game, supply chain coordination can be achieved by the properly-designed buy-back and lost-sales cost-sharing contracts. In contrast, if only the buy-back or lost-sales cost-sharing contract is involved, coordination of the supply chain cannot be achieved. For instance, when lost-sales cost-sharing contract is not considered, the value of the term ϕ_i $(i = 1, \ldots, n)$ becomes zero. Then the conditions given in (19) are reduced to $zv = 0$ and $z = (w_k - c_k)/v_k$. From the former condition, we find that $v = 0$. This makes the second condition unsatisfied due to $v_k = 0$. On the other hand, if we just incorporate the lost sales cost-sharing contract, then the first condition in (19) cannot be satisfied since $(1 - z)\phi u \neq 0$. Additionally, we observe from Theorem 5 that there must exist multiple feasible buy-back and lost-sales cost-sharing contracts which achieve supply chain coordination. For example, we can set the values of ϕ as

$$\phi_k = \sum_{j=1}^{k-1} \varepsilon_j + \sum_{j=k+1}^{n} \varepsilon_j \quad \text{and} \quad \phi_i = \frac{1}{c}\left(c_i + \frac{c_k}{n-1}\right) - \varepsilon_i, \quad \text{for } i = 1, \ldots, k-1, k+1, \ldots, n,$$

where ε_i $(i \neq k)$ is sufficiently small number so that all inequalities in (20) are satisfied. ♦

For the simultaneous-move game, when the random demand is dependent of price in the additive form, we use a buy-back and lost-sales cost-sharing contract design specified by (21) and (22) to achieve supply chain coordination, under which the equilibrium solution is identical to the globally optimal solution. From (21), we find that the supplier S_k with the smallest unit production cost is also the one with smallest $z_k = \min_{i=1,\ldots,n} z_i$. Thus, from Theorem 3, all suppliers use the same production quantity which is determined in terms of z_k. In this paper, the supplier S_k with the smallest unit production cost is called the "critical" supplier, and the others are called "non-critical."

For the lost-sales cost-sharing contract, the critical supplier S_k has the lowest production cost, i.e., the value of this supplier's component has the least value in the final product. This means that the critical supplier should have the least responsibility for sharing the cost of lost sales, thus we set $\phi_k = 0$.

For the simultaneous-move game, we list five steps to be used for our contract design that achieves supply chain coordination:

Step 1: Find the globally optimal solution (p^*, q^*) from Theorem 4;

Step 2: Compute $z = (p^* - m + u - c)/(p^* - m + u)$;

Step 3: Select the critical supplier S_k who has the minimum unit production cost;

Step 4: Determine supplier S_k's wholesale price (w_k), buyback price (v_k) and percentage of the underage cost absorbed by this supplier (ϕ_k) as follows: $w_k = (1 + z)c_k$, $v_k = c_k$, and $\phi_k = 0$;

Step 5: Determine the other suppliers' wholesale price, buyback price and percentage of the underage cost absorbed by them as follows: for $i = 1, \dots, k-1, k+1, \dots, n$,

$$\begin{cases} w_i = \left(\frac{1-z}{cz}u + 1\right) c_i + \left(\frac{1-z}{cz}u - 1\right) \frac{c_k}{n-1}, \\ v_i = \frac{1-z}{cz}uc_i + \left(\frac{1-z}{cz}u - 1\right) \frac{c_k}{n-1}, \\ \phi_i = \frac{1}{c}\left(c_i + \frac{c_k}{n-1}\right). \end{cases}$$

Under the properly-designed contracts specified in Theorem 5, we can maximize the system-wide profit but cannot assure that each supply chain member individually benefits from the buy-back and lost-sales cost-sharing contracts. Next, we examine whether or not each member's individual profit under the proper contracts is higher than that without the buy-back and lost-sales cost-sharing contracts. Note that, if the buy-back contract (w, v) is not involved, then supplier S_i $(i = 1, 2, \dots, n)$ does not need to buy unused components back from the manufacturer and thus the parameter $v_i = 0$, for $i = 1, 2, \dots, n$. If the lost-sales cost-sharing contract is not involved, then each supplier's share of shortage cost is zero, i.e., $\phi_i = 0$, for $i = 1, 2, \dots, n$.

When $v_i = \phi_i = 0$, for $i = 1, 2, \dots, n$, we can reduce supplier S_i's random profit in (2) and the manufacturer's random profit in (5), respectively, to

$$\Pi_{S_i} = w_i Q + s_i(q_i - Q_{-i})^+ - c_i q_i, \tag{23}$$

$$\Pi_M = (p - m)\min[Q, D(p, \varepsilon)] - wQ - u[D(p, \varepsilon) - Q]^+. \tag{24}$$

Without the buy-back and lost-sales cost-sharing contracts, the manufacturer must find optimal retail price to maximize its expected profit $E(\Pi_M)$ and supplier S_i must make an optimal quantity decision to maximize its expected profit $E(\Pi_{S_i})$.

Theorem 6 If the buy-back and lost-sales cost-sharing contracts are not involved for the additive demand case, then the Nash equilibrium for the simultaneous-move game can be uniquely determined as:

$$p^N = \frac{a + mb + B - \int_A^B F(x)dx}{2b} \quad \text{and} \quad q_i^N = \frac{a - mb + B + \int_A^B F(x)dx}{2},$$

for $i = 1, \dots, n$ and $q^N = q_1^N = \cdots = q_n^N$. ◀

We learn from Remark 1 that, without the buy-back and lost-sales cost-sharing contracts, the supply chain cannot be coordinated because, for any value of w_i $(i = 1, 2, \dots, n)$, the Nash equilibrium is always *not* identical to the globally-optimal solution.

Theorem 7 If the buy-back and lost-sales cost-sharing contracts are not involved for the additive demand case, then, for any value of w_i $(i = 1, 2, \dots, n)$, the total profit in terms of the Nash equilibrium given in Theorem 6 is always smaller than that in terms of the globally-optimal solutions given in Theorem 4. ◀

We learn from Theorem 7 that the supply chain-wide performance is worse off if the buy-back and lost-sales cost-sharing contracts are not involved. We denote the supplier S_i's and the manufacturer's expected profits without the contractual mechanism by $E(\tilde{\Pi}_{S_i})$ and $E(\tilde{\Pi}_M)$, respectively. Note that, under the properly-designed contracts given in Theorem 5, S_i's and the manufacturer's expected profits in terms of (p^N, q^N)—denoted by $E(\Pi_{S_i}^N)$ and $E(\Pi_M^N)$—are respectively equal to $E(\Pi_{S_i}^*)$ and $E(\Pi_M^*)$, which represent S_i's and the manufacturer's expected profits in terms of the globally-optimal solution (p^*, q^*). That is, $E(\Pi_{S_i}^N) = E(\Pi_{S_i}^*)$, $i = 1, 2, \ldots, n$, and $E(\Pi_M^N) = E(\Pi_M^*)$. According to Theorem 7, we have

$$\sum\nolimits_{i=1}^{n} E(\tilde{\Pi}_{S_i}) + E(\tilde{\Pi}_M) < \sum\nolimits_{i=1}^{n} E(\Pi_{S_i}^*) + E(\Pi_M^*) = \sum\nolimits_{i=1}^{n} E(\Pi_{S_i}^N) + E(\Pi_M^N),$$

which cannot assure that $E(\Pi_{S_i}^*) > E(\tilde{\Pi}_{S_i})$ [or, $E(\Pi_{S_i}^N) > E(\tilde{\Pi}_{S_i})$] and $E(\Pi_M^*) > E(\tilde{\Pi}_M)$ [or, $E(\Pi_M^N) > E(\tilde{\Pi}_M)$]. This means that, even though the system-wide profit can be increased under the properly-designed contracts, n suppliers and the manufacturer may not all individually benefit from the contractual mechanism. If $E(\Pi_{S_i}^*) < E(\tilde{\Pi}_{S_i})$ or $E(\Pi_M^*) < E(\tilde{\Pi}_M)$, then supplier S_i or the manufacturer would lose an incentive to cooperate with the others for the buy-back and lost-sales cost-sharing contracts. Thus, we need to allow all supply chain members to *fairly* share the system-wide profit surplus

$$\gamma \equiv \left[\sum\nolimits_{i=1}^{n} E(\Pi_{S_i}^*) + E(\Pi_M^*)\right] - \left[\sum\nolimits_{i=1}^{n} E(\tilde{\Pi}_{S_i}) + E(\tilde{\Pi}_M)\right]. \tag{25}$$

We assume that supplier S_i receives $\gamma_{S_i} > 0$, for $i = 1, 2, \ldots, n$, and the manufacturer receives $\gamma_M > 0$, such that $\sum_{i=1}^{n} \gamma_{S_i} + \gamma_M = \gamma$. After receiving their shares, supplier S_i's and the manufacturer's eventual profits are $E(\tilde{\Pi}_{S_i}) + \gamma_{S_i}$ $(i = 1, 2, \ldots, n)$ and $E(\tilde{\Pi}_M) + \gamma_M$, respectively. This means that, after sharing γ, all members' profits are higher than those without the contractual mechanism.

In order to *fairly* determine the allocation of the profit surplus γ for the n suppliers, one-manufacturer supply chain, we use the "Shapley value" concept to calculate γ_{S_i} $(i = 1, 2, \ldots, n)$ and γ_M. Shapley value, developed by Shapley [16], is an important solution concept for cooperative games, which, for our paper, provides a unique scheme for allocating the profit surplus γ among $(n+1)$ players including n suppliers and the manufacturer. The unique Shapley values $\boldsymbol{\gamma} = (\gamma_{S_1}, \ldots, \gamma_{S_n}; \gamma_M)$ are determined by $\gamma_j = \{\sum_{j \in T}(|T| - 1)![n + 1 - |T|]![v(T) - v(T - j)]\}/(n+1)!$, for $j = S_1, \ldots, S_n, M$, where T denotes a coalition in which some supply chain members cooperate to jointly make their decisions, $v(\cdot)$ denotes the profit surplus joint achieved by all cooperative members in a coalition, and $|T|$ is the size of T. Note that, in the assembly supply chain, all suppliers sell *complementary* components to the manufacturer who assembles and sells final products. This implies that the system-wide profit would be zero if any member in such a supply chain leaves. Using this fact, we can compute the Shapley value-based allocation of the profit surplus γ as shown in the following theorem.

Theorem 8 Shapley value suggests that the system-wide profit surplus γ should be equally allocated among n suppliers and the manufacturer, i.e., $\gamma_{S_1} = \gamma_{S_2} = \cdots = \gamma_{S_n} = \gamma_M = \gamma/(n+1)$. As a result, n suppliers' and the manufacturer's profits after the allocation are, respectively, computed as $E(\tilde{\Pi}_{S_i}) + \gamma/(n+1)$ $(i = 1, 2, \ldots, n)$ and $E(\tilde{\Pi}_M) + \gamma/(n+1)$. ◄

In conclusion, we find that, in order to coordinate the supply chain for the simultaneous-move game with the additive demand form, it is important to consider the following two issues: (i) We should design the buy-back and lost-sale cost-sharing contracts as in Theorem 5 to assure that the Nash equilibrium and the global solution are identical; (ii) we should fairly allocate the system-wide profit surplus γ as in Theorem 8 to assure that all supply chain members are better off than without the contractual mechanism.

2.2.3 Design of Buy-Back and Lost-Sales Cost-Sharing Contracts for the Leader-Follower Game

We now consider the leader-follower game, and design a pair of proper buyback and lost-sales cost-sharing contracts to achieve supply chain coordination. That is, under the properly-designed contracts, the Stackelberg equilibrium $(p^S, q_1^S, q_2^S, \ldots, q_n^S)$ given in Theorem 3 is identical to the globally-optimal solution $(p^*, q_1^*, q_2^*, \ldots, q_n^*)$ given in Theorem 4.

Similar to Section 2.2.2, we use Theorems 3 and 4 to re-write the Stackelberg equilibrium (p^S, q^S) and the global solution (p^*, q^*) as

$$\begin{aligned} \int_A^{q^S - y(p^S)} F(x)dx + (p^S - m - w)b &= q^S, &(26)\\ q^S &= y(p^S) + F^{-1}(z_k), &(27) \end{aligned}$$

and

$$\begin{aligned} \int_A^{q^* - y(p^*)} F(x)dx + (p^* - m - c)b &= q^*, &(28)\\ q^* &= y(p^*) + F^{-1}(z), &(29) \end{aligned}$$

where $z = (p^* - m + u - c)/(p^* - m + u)$, as defined in Section 2.2.2.

It is important to note that the Stackelberg equilibrium will be identical to the globally optimal solution if and only if (26) is identical to (28) and (27) is identical to (29). Equating the stated equations and simplifying, the conditions that will assure $p^S = p^*$ and $q^S = q^*$ are found as

$$w = c, \qquad \text{and} \qquad z = \frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}, \tag{30}$$

where the chosen values of parameters which satisfy the above conditions also meet the five requirements (20).

Theorem 9 For the leader-follower game with the additive demand form (1), supply chain coordination can be achieved by a pair of the properly-designed buy-back and lost-sales cost-sharing contract with the following parameter values:

1. If $u \le c$, then the proper contracts can be designed as

$$v_i = (1-z)u\frac{c_i}{c}, \quad \phi_i = z\frac{c_i}{c}, \quad w_i = c_i,$$

for $i = 1, 2, \ldots, n$.

2. If $u > c$, then the proper contracts can be designed as

$$v_i = (1 - z)c_i, \quad \phi_i = z\frac{c_i}{u}, \quad w_i = c_i,$$

for $i = 1, 2, \ldots, n$. ◀

We learn from Theorem 9 that, if all suppliers make their production decisions after the manufacturer makes its pricing decision, then all suppliers' wholesale prices must be equal to their production costs in order to induce supply chain coordination. Hence, for the leader-follower game, each supplier's unit profit is zero; see, Bernstein and Federgruen [2] for a similar result. As we discussed in Section 2.2.2, all supply chain members should be better off than without the contractual mechanism; otherwise, they may lose the incentive to cooperate for supply chain coordination.

Next, we compute the Stackelberg equilibrium when the buy-back and lost-sales cost-sharing contracts are not involved, and examine whether or not the supply chain can be coordinated without such contracts.

Theorem 10 If the buy-back and lost-sales cost-sharing contracts are not involved for the additive demand case, then the Stackelberg equilibrium for the leader-follower game can be uniquely determined as:

$$p^S = \frac{a + b(m + w) + \mu}{2b} \quad \text{and} \quad q_i^S = \frac{a - b(m + w) - \mu + 2B}{2},$$

for $i = 1, \ldots, n$ and $q_1^S = \cdots = q_n^S$.

Moreover, we find that the supply chain cannot be coordinated if the buy-back and lost-sales cost-sharing contracts are not involved. This means that, for any value of w_i $(i = 1, 2, \ldots, n)$, the total profit in terms of the Stackelberg equilibrium is always smaller than that in terms of the globally-optimal solutions given in Theorem 4. ◀

We can use Theorem 9 to achieve supply chain coordination under proper contracts but cannot assure that all supply chain members are better off than without the contracts. As Theorem 10 indicates, without the buyback and shortage contracts, the total profit in terms of the Stackelberg equilibrium is always smaller than that in terms of the globally-optimal solutions given in Theorem 4. Similar to Section 2.2.2, in order to entice all members to cooperate for supply chain coordination, we should use (25) to compute the expected profit surplus generated under the proper contracts given in Theorem 9, and use Theorem 8 to calculate the allocations to n suppliers and the manufacturer.

In online Appendix B, we provide two numerical examples to illustrate our analysis for the additive demand case.

3 Non-Cooperative Equilibria and Supply Chain Coordination with Price-Dependent Random Demand: Multiplicative Form

In this section, the manufacturer faces the random price-dependent demand in a multiplicative form. Note that the difference between our analyses in this section and Section 2 is the form of the demand function. Thus, in order not to be repetitive, we only present Nash and Stackelberg equilibria, globally-optimal solution, and contract design in what follows.

As Petruzzi and Dada [14] discussed, the multiplicative demand function in the newsvendor context is commonly formulated as

$$D(p,\varepsilon) = y(p)\varepsilon, \tag{31}$$

where the deterministic term $y(p)$ and the error term ε taking values in the range $[A,B]$ as defined in Table 1. Similar to Section 2.1.2, it is reasonable to assume that the error term has a lower bound $A>0$ and an upper bound $B<\infty$. The assumption was also commonly made in previous publications; see, for example, Petruzzi and Dada [14]. As in Section 2, the demand function (31) is "common knowledge" for all suppliers and the manufacturer; thus, all suppliers know the demand range $[Ay(p), By(p)]$, and determine their quantities subject to this range, i.e., $Ay(p) \le q_i \le By(p)$, $i=1,\ldots,n$.

The next theorem gives the Nash equilibrium for the simultaneous-move game and the Stackelberg equilibrium for the leader-follower game.

Theorem 11 The Nash equilibrium for the multiplicative case should be obtained by solving the following equation set:

$$\begin{aligned} & F^{-1}(z_k)(1-z_k) + \left[1 - (p^N - m - v)\frac{b}{p^N}\right]\mu \\ = \; & \left\{1 - [p^N - m + (1-\phi)u - v]\frac{b}{p^N}\right\}\int_{F^{-1}(z_k)}^{B} xf(x)dx, \end{aligned} \tag{32}$$

$$q^N \equiv q_1^N = \cdots = q_n^N = y(p^N)F^{-1}(z_k), \tag{33}$$

where $z_k = (w_k - c_k + \phi_k u)/(v_k + \phi_k u)$ with the index k defined by (4).

The Stackelberg equilibrium for the leader-follower game should be obtained by solving the following equation set:

$$\begin{aligned} & F^{-1}(z_k)[1-z_k] + \left[1 - (p^S - m - v)\frac{b}{p^S}\right]\mu \\ = \; & \{[p^S - m + (1-\phi)u - v](1-z_k) + (v-w)\}\frac{b}{p^S}F^{-1}(z_k) \\ & + \left\{1 - [p^S - m + (1-\phi)u - v]\frac{b}{p^S}\right\}\int_{F^{-1}(z_k)}^{B} xf(x)dx, \end{aligned} \tag{34}$$

$$q_1^S = q_2^S = \ldots = q_n^S = y(p^S)F^{-1}(z_k). \quad \blacktriangleleft \tag{35}$$

Next, we compute the globally-optimal pricing and production decisions that maximize the chainwide profit.

Theorem 12 For the multiplicative case, the globally optimal price p^* and production quantity q^* satisfy the conditions

$$\begin{aligned} & F^{-1}(\xi(p^*))[1-\xi(p^*)] + \left[1-(p^*-m)\frac{b}{p^*}\right]\mu \\ = & \left[1-(p^*-m+u)\frac{b}{p^*}\right]\int_{F^{-1}(\xi(p^*))}^{B} xf(x)dx, \quad (36) \\ q^* = & \; y(p^*)F^{-1}(\xi(p^*)), \quad (37) \end{aligned}$$

where $\xi(p^*) = (p^* - m + u - c)/(p^* - m + u)$, as defined in (14). ◀

Next theorem indicates the impacts of the parameter b in the multiplicative demand function (31) on the Nash and Stackelberg equilibria and the globally-optimal solution.

Theorem 13 As the value of b in (31) increases, both Nash and Stackelberg equilibrium prices (p^N and p^S) decrease, and the globally-optimal solution p^* also decreases. ◀

Our properly-designed contracts for both simultaneous-move and leader-follower games are given in the next theorem.

Theorem 14 For both simultaneous-move and leader-follower games with the multiplicative demand form, supply chain coordination can be achieved by a pair of the properly-designed buy-back and lost-sales cost-sharing contracts. The proper contract designs for the two games, which are given in Table 2, depend on the value of $\kappa \equiv \lambda u/[c(\mu - \lambda)]$ with

$$\lambda \equiv \begin{cases} \int_{q^*/y(p^*)}^{B} xf(x)dx, & \text{for the simultaneous-move game,} \\ \int_{q^*/y(p^*)}^{B} [x - q^*/y(p^*)]f(x)dx, & \text{for the leader-follower game.} \end{cases}$$ ◀

Theorem 14 demonstrates that, for both simultaneous-move and leader-follower games, this assembly supply chain can be coordinated by a pair of proper buyback and lost-sales cost-sharing contracts. Different from the additive case in Section 2, for the multiplicative case we have to consider a specific condition [i.e., $\kappa \leq 1$] and present different properly-designed contracts when the condition is satisfied and not satisfied.

Next, similar to Section 2, we compute the Nash and Stackelberg equilibria when the buy-back and lost-sales cost-sharing contracts are not involved.

Theorem 15 Without the buy-back and lost-sales cost-sharing contracts, the Nash and Stackelberg equilibria are obtained as follows:

$$\begin{aligned} p^N &= bm/(b-1), \quad q_i^N = By(p^N),\ i = 1, \ldots, n; \\ p^S &= b(m\mu + wB)/[(b-1)\mu], \quad q_i^S = By(p^S),\ i = 1, \ldots, n. \end{aligned}$$

(i) $\kappa \le 1$		
Supplier	Supplier S_k	Supplier S_i $i = 1, \ldots, k-1, k+1, \ldots, n$
Contract Design	$w_k = (1 + z\kappa)\, c_k$, $v_k = \kappa c_k$, $\phi_k = 0.$	$w_i = v_i + c_i = (\kappa + 1)\, c_i$, $v_i = \kappa c_i$, $\phi_i = \frac{1}{c}\left(c_i + \frac{c_k}{n-1}\right).$
(ii) $\kappa \ge 1$		
Supplier	Supplier S_k	Supplier S_i $i = 1, \ldots, k-1, k+1, \ldots, n$
Contract Design	$w_k = (1 + z)c_k$, $v_k = c_k$, $\phi_k = 0.$	$w_i = v_i + c_i = (\kappa + 1)\, c_i + (\kappa - 1)\, \frac{c_k}{n-1}$, $v_i = \kappa c_i + (\kappa - 1)\, \frac{c_k}{n-1}$, $\phi_i = \frac{1}{c}\left(c_i + \frac{c_k}{n-1}\right).$
Note that S_k is chosen as the supplier with the smallest unit production cost, i.e., $c_k = \min_{i=1,\ldots,n} c_i$.		

Table 2: The set of contract parameters for the multiplicative case.

We find that the total profit in terms of the Nash or Stackelberg equilibrium is always smaller than that in terms of the globally-optimal solutions given in Theorem 12. ◀

In order to entice all members to cooperate for supply chain coordination, we use (25) to compute the expected profit surplus γ, and use Theorem 8 to calculate the allocations to n suppliers and the manufacturer.

In online Appendix C, we provide two numerical examples to illustrate our analysis for the multiplicative demand case.

4 Conclusions and Recommendations for Further Research

In this paper we considered an assembly supply chain where multiple suppliers produce complementary components that are used by a manufacturer who assembles the final product and sells them directly to a market. The single period demand for the final product is random and depends on the retail price chosen by the manufacturer.

The suppliers and the manufacturer have an agreement whereby each supplier buys back the unsold components and absorbs a portion of the lost-sales cost. Accordingly, we incorporate the buy-back and lost sales cost-sharing contracts into our inventory-related game model where we assume a price-sensitive random demand in (i) additive, and (ii) multiplicative form. For each case, we develop a pair of properly-designed lost-sales cost-sharing contracts to induce supply chain coordination. To determine the contracts, we first consider the situation where the suppliers and the manufacturer do not cooperate and where the suppliers and the manufacturer determine their production quantities and retail price, respectively. We derive the Nash equilibrium for the simultaneous-move game, and compute the Stackelberg equilibrium for the leader-follower game.

Next, we examine whether or not the assembly supply chain can be coordinated. We show that, for both simultaneous-move and leader-follower games, the supply chain coordination can always be achieved by a pair of properly-designed buy-back and lost-sales cost-sharing contracts. Under the properly-designed contracts, each supplier adopts the equilibrium production quantity and the manufacturer chooses the equilibrium price, and the system-wide profit is maximized as well.

Since in this paper the demand is assumed to depend on the retail price only, we may relax this assumption in future and analyze assembly supply chains where demand depends on price, quality and some other factors. In another future research direction, we may consider the case where the manufacturer sets the buy-back and shortage parameters to increase its own profits while allowing the suppliers to gain profits so that they are willing to keep the business with the manufacturer. This would be modeled as a two-stage problem in which the manufacturer first sets the contract parameters and all players (i.e., the manufacturer and its suppliers) then find the Nash equilibrium.

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Online Appendices

"Game-Theoretic Analyses of Decentralized Assembly Supply Chains: Non-Cooperative Equilibria vs. Coordination with Cost-Sharing Contracts"

Mingming Leng and Mahmut Parlar

Appendix A Proofs

Proof of Theorem 1. Supplier S_i's profit function in (2) is evaluated by comparing this supplier's production quantity q_i with the minimum production quantity Q_{-i} of the other suppliers: If $Q_{-i} \leq q_i$, then $Q = Q_{-i}$; otherwise, if $q_i \leq Q_{-i}$, then $Q = q_i$ in which case S_i receives no salvage value. As discussed in Section 2.1.2, the production quantity q_i will fall in the range $[y(p)+A, y(p)+B]$, hence, $Q_{-i} \in [y(p)+A, y(p)+B]$. In order to find the optimal production quantity for S_i, we examine the profit function for two different cases: (i) $Q_{-i} \leq q_i \leq y(p)+B$, and (ii) $y(p)+A \leq q_i \leq Q_{-i}$.

(i) $Q_{-i} \leq q_i \leq y(p)+B$: In this case, $Q = Q_{-i}$ and S_i's profit function (2) is reduced to

$$\Pi_{S_i} = (w_i - s_i)Q_{-i} + q_i(s_i - c_i) - v_i[Q_{-i} - D(p,\varepsilon)]^+ - \phi_i u[D(p,\varepsilon) - Q_{-i}]^+.$$

Hence, the expected profit is

$$E(\Pi_{S_i}) = (w_i - s_i)Q_{-i} + q_i(s_i - c_i) - v_i E[Q_{-i} - D(p,\varepsilon)]^+ - \phi_i u E[D(p,\varepsilon) - Q_{-i}]^+,$$

which shows that $E(\Pi_{S_i})$ is decreasing in q_i due to $s_i < c_i$. As a result,

$$q_i^* = Q_{-i}, \text{ for } Q_{-i} \leq q_i \leq y(p)+B.$$

(ii) $y(p)+A \leq q_i \leq Q_{-i}$: In this case S_i's profit function (2) becomes

$$\Pi_{S_i} = (w_i - c_i)q_i - v_i[q_i - D(p,\varepsilon)]^+ - \phi_i u[D(p,\varepsilon) - q_i]^+,$$

and the expected profit is

$$E(\Pi_{S_i}) = (w_i - c_i)q_i - v_i \int_A^{q_i - y(p)} [q_i - y(p) - x] f(x)\, dx - \phi_i u \int_{q_i - y(p)}^{B} [y(p) + x - q_i] f(x)\, dx.$$

Taking the first- and second-order derivatives w.r.t. q_i, we have

$$\begin{aligned} \frac{dE(\Pi_{S_i})}{dq_i} &= w_i - c_i + \phi_i u - (v_i + \phi_i u)F(q_i - y(p)), \\ \frac{d^2E(\Pi_{S_i})}{dq_i^2} &= -(v_i + \phi_i u) f(q_i - y(p)) < 0, \end{aligned}$$

which shows that S_i's expected profit $E(\Pi_{S_i})$ is a strictly concave function of q_i. Thus, we can solve $dE(\Pi_{S_i})/dq_i = 0$ and obtain the unconstrained solution that maximizes $E(\Pi_{S_i})$ as

$$q_i^0 = y(p) + F^{-1}\left(\frac{w_i - c_i + \phi_i u}{v_i + \phi_i u}\right).$$

Furthermore, since $0 < w_i - c_i \leq v_i$ as discussed in Section 2.1, we have

$$A = F^{-1}(0) < F^{-1}\left(\frac{w_i - c_i + \phi_i u}{v_i + \phi_i u}\right) \leq F^{-1}(1) = B.$$

Now, for this case, taking into account the constraint $q_i \leq Q_{-i}$, we find the optimal solution (i.e., the supplier S_i's best response production decision) as

$$q_i^B = \min\left[y(p) + F^{-1}\left(\frac{w_i - c_i + \phi_i u}{v_i + \phi_i u}\right), Q_{-i}\right].$$

Combining our analysis for Cases (i) and (ii), we find that, for given contract parameters $(\mathbf{v}, \boldsymbol{\phi})$, the manufacturer's pricing decision p, and other suppliers' production quantities Q_{-i}, the ith supplier's optimal best response production quantity q_i^B is determined as

$$q_i^B = \min\left[y(p) + F^{-1}\left(\frac{w_i - c_i + \phi_i u}{v_i + \phi_i u}\right), Q_{-i}\right], \tag{38}$$

which implies that supplier S_i's best response production quantity is no more than the minimum production quantity of other suppliers, i.e., $q_i^B \leq Q_{-i}$.

Our above discussion shows that each supplier's best response production decision can be calculated as a function of the other suppliers' decisions and the manufacturer's retail price. Next we provide an expression for the exact value of the suppliers' production quantities.

We first demonstrate that all suppliers' equilibrium production quantities are equal. Suppose that, for a fixed manufacturer's price p, suppliers S_i and S_j have different best responses, that is, $q_i^B \neq q_j^B$. We then have

$$\begin{aligned} q_i^B &\leq Q_{-i}^B = \min\{q_1^B, \ldots, q_{i-1}^B, q_{i+1}^B, \ldots, q_n^B\} \leq q_j^B, \\ q_j^B &\leq Q_{-j}^B = \min\{q_1^B, \ldots, q_{j-1}^B, q_{j+1}^B, \ldots, q_n^B\} \leq q_i^B, \end{aligned}$$

implying that $q_i^B = q_j^B$, which contradicts our assumption, hence, we must have $q_1^B = q_2^B = \cdots = q_n^B$.

We assume that the supplier with the smallest ratio of the right-hand-side term in (4) is S_k which results in

$$F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right) = \min_{i=1,\ldots,n}\left[F^{-1}\left(\frac{w_i - c_i + \phi_i u}{v_i + \phi_i u}\right)\right]. \tag{39}$$

We have proved that supplier S_k's best response production quantity is

$$q_k^B = \min\left[y(p) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right), Q_{-k}^B\right].$$

From (39), we find that

$$y(p) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right) \leq Q_{-k}^B.$$

Hence, each supplier's best response is

$$q_1^B = \cdots = q_n^B = y(p) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right),$$

and this theorem proves. ■

Proof of Theorem 2. We first consider the manufacturer's best response price for the simultaneous-move game. We begin by showing that the expected profit $E(\Pi_M)$ is a unimodal function of the price p. Taking the first- and second-order derivatives of $E(\Pi_M)$ in (6) w.r.t. p, we have

$$\frac{dE(\Pi_M)}{dp} = Q + (1-\phi)ub - \int_A^{Q-y(p)} F(x)dx - [p - m + (1-\phi)u - v]bF(Q - y(p)),$$

and

$$\frac{d^2E(\Pi_M)}{dp^2} = -2bF(Q - y(p)) - [p - m + (1-\phi)u - v]b^2 f(Q - y(p)). \tag{40}$$

As $-2bF(Q - y(p)) < 0$, we will investigate the sign of the second term in (40). Since $b^2 f(Q - y(p)) > 0$, it is sufficient to show that $[p - m + (1-\phi)u - v] > 0$ to prove the nonnegativity of the second term. Consider the point(s) at which $dE(\Pi_M)/dp = 0$. We have,

$$\begin{aligned}
[p - m + (1-\phi)u - v]bF(Q - y(p)) &= Q + (1-\phi)ub - \int_A^{Q-y(p)} F(x)\, dx \\
&\geq Q + (1-\phi)ub - [Q - y(p) - A] \\
&\geq (1-\phi)ub + y(p) + A > 0.
\end{aligned}$$

Since $bF(Q - y(p)) > 0$, we have $[p - m + (1-\phi)u - v] > 0$ and $d^2E(\Pi_M)/dp^2 < 0$ for any price satisfying $dE(\Pi_M)/dp = 0$. This implies that the manufacturer's expected profit is a unimodal function of the unit retail price p with a unique maximizing value p^B.

We now consider the manufacturer's best-response price p^B for the leader-follower game. From Theorem 1, we have found all suppliers' best production quantities as

$$q_1^B = \cdots = q_n^B = y(p) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right),$$

which means that the manufacturer receives

$$Q = \min(q_1^B, q_2^B, \ldots, q_n^B) = y(p) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right). \tag{41}$$

In order to obtain the manufacturer's best response price p^B, we substitute (41) into (6), and find

$$\begin{aligned}
E(\Pi_M) = {} & (p - m - v)[y(p) + \mu] + (v - w)\left[y(p) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right)\right] \\
& + [p - m + (1-\phi)u - v] \\
& \times \int_{F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right)}^{B} \left[F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right) - x\right] f(x)dx.
\end{aligned} \tag{42}$$

The first- and second-order derivatives of $E(\Pi_M)$ in (42) w.r.t. p are calculated as follows:

$$\begin{aligned}
\frac{\partial E(\Pi_M)}{\partial p} &= a - 2bp + b(m+w) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right) \\
&\quad - \int_A^{F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right)} F(x)dx, \\
\frac{\partial^2 E(\Pi_M)}{\partial p^2} &= -2b < 0,
\end{aligned}$$

which implies that $E(\Pi_M)$ in (42) is strictly concave in p. Equating $\partial E(\Pi_M)/\partial p$ to zero and solving the resulting equation for p, we can find the manufacturer's best response p^B as

$$p = \frac{1}{2b}\left[a + b(m+w) + F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right) - \int_A^{F^{-1}\left(\frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}\right)} F(x)dx\right],$$

which, using (41), can be re-written as

$$\int_A^{Q-y(p)} F(x)dx + (p - m - w)b = Q,$$

as shown in (8). ∎

Proof of Theorem 3. We first consider the simultaneous-move game. In order to find the Nash equilibrium $(p^N, q_1^N, \ldots, q_n^N)$, we recall from Theorem 1 that $q \equiv q_1^B = \cdots = q_n^B$. As Theorem 2 indicates, the manufacturer's best response price p^B is found as the solution of the following nonlinear equation

$$\begin{aligned}
\int_A^{q-y(p)} F(x)dx + [p - m + (1-\phi)u - v]bF(q - y(p)) &= q + (1-\phi)ub, &(43)\\
q &= y(p) + F^{-1}(z_k). &(44)
\end{aligned}$$

From the best response function of each supplier, we have $F(q - y(p)) = z_k$. Hence, we reduce the equation in (43) to

$$\int_A^{F^{-1}(z_k)} F(x)dx + [p - m + (1-\phi)u - v]bz_k = q + (1-\phi)ub,$$

and solve it for p to find

$$\begin{aligned}
p &= m - (1-\phi)u + v + \frac{1}{bz_k}\left[q + (1-\phi)ub - \int_A^{F^{-1}(z_k)} F(x)\,dx\right] \\
&= m + v + \left(\frac{1}{z_k} - 1\right)(1-\phi)u + \frac{1}{bz_k}\left[q - \int_A^{F^{-1}(z_k)} F(x)dx\right]. \qquad (45)
\end{aligned}$$

Substituting the expression for p into (44), we obtain the equilibrium production quantity q^N of each supplier as (10). Finally, substituting q^N of (10) into (45), we obtain the expression for p^N in (9).

Next, we find the Stackelberg equilibrium when the manufacturer and the suppliers act the leader and the followers, respectively. We learn from the proof of Theorem 2 that the Stackelberg price p^S of the manufacturer can be computed as in (11). Using Theorem 1 we find that, for the leader-follower game,

$$q_i^S = y(p^S) + F^{-1}(z_k) = a - bp^S + F^{-1}(z_k),\ i = 1, 2, \ldots, n.$$

Substituting (11) into the above equation gives each supplier's Stackelberg equilibrium q_i^S in (12). ■

Proof of Theorem 4. Partially differentiating $E(\Pi)$ w.r.t. p and q, we have

$$\begin{aligned}
\frac{\partial E(\Pi)}{\partial p} &= [y(p) + \mu] + bu + \int_{q-y(p)}^{B} [q - y(p) - x] f(x)\ dx - b(p - m + u)F(q - y(p)), \\
\frac{\partial E(\Pi)}{\partial q} &= -c + (p - m + u)[1 - F(q - y(p))].
\end{aligned}$$

The second-order partial derivatives are computed as

$$\begin{aligned}
\frac{\partial^2 E(\Pi)}{\partial p^2} &= -2bF(q - y(p)) - b^2(p - m + u)f(q - y(p)) < 0, \\
\frac{\partial^2 E(\Pi)}{\partial q^2} &= -(p - m + u)f(q - y(p)) < 0.
\end{aligned}$$

Thus, for fixed q, $E(\Pi)$ is strictly concave in p; and for fixed p, it is strictly concave in q. Even though we have computed the Hessian, the complicated nature of the problem made it difficult to analyze the Hessian's negative definiteness. Thus, we cannot determine the concavity or the unimodality of the function $E(\Pi)$. However, as Petruzzi and Dada [14, Theorem 1] have shown, the finite optimal solution (p^*, q^*) for a single-period profit model with random price-dependent demand (where the expected profit is not necessarily concave or unimodal) can be found by solving

$$\begin{aligned}
q + bu - \int_{A}^{q-y(p)} F(x)dx - b(p - m + u)F(q - y(p)) &= 0, && (46) \\
-c + (p - m + u)[1 - F(q - y(p))] &= 0. && (47)
\end{aligned}$$

From (47), we find

$$F(q - y(p)) = \frac{p - m + u - c}{p - m + u},$$

or,

$$q(p) = y(p) + F^{-1}\left(\frac{p - m + u - c}{p - m + u}\right). \tag{48}$$

Substituting (48) into (46) gives

$$2bp + \int_{A}^{F^{-1}(\xi(p))} F(x)dx - F^{-1}(\xi(p)) = a + b(m + c), \tag{49}$$

where $\xi(p)$ is as given in the statement of the theorem. Since $\partial^2 E(\Pi)/\partial p^2 < 0$, the solution of (49) should be unique, thus giving the globally-optimal retail price p^* which can then be used to compute the globally optimal q^* from (48). $\blacksquare$

Proof of Theorem 5. We need to show that the values of parameters specified in (21) and (22) satisfy (19) and (20). It is easy to see that conditions in (19) are satisfied. Now we examine whether the values of the parameters can satisfy (20).

1. $w_i - c_i \le v_i \le w_i, \quad i = 1, \dots n$: Since $zc_k \le c_k \le (1+z)c_k$, we have $w_k - c_k \le v_k \le w_k$. Furthermore, due to $w_i = v_i + c_i$, we have $w_i - c_i \le v_i \le w_i$, $i = 1, \dots, k-1, k+1, \dots, n$.
2. $0 \le \phi_i \le 1, \quad i = 1, \dots, n$: From (21), we have $\phi_k = 0$. Since $c = \sum_{j=1}^n c_j > c_i + c_k > c_i + c_k/(n-1)$, we find that $0 \le \phi_i \le 1$, $i = 1, \dots, k-1, k+1, \dots, n$.
3. $\sum_{i=1}^n \phi_i \le 1$: In such a contract design, we have,

$$\begin{aligned}\sum_{i=1}^n \phi_i &= \phi_1 + \cdots + \phi_{k-1} + \phi_k + \phi_{k+1} + \cdots + \phi_n \\ &= \frac{(c_1 + \cdots + c_{k-1} + c_{k+1} + \cdots + c_n) + c_k}{c} + \phi_k \\ &= 1.\end{aligned}$$

4. $\dfrac{w_i - c_i + \phi_i u}{v_i + \phi_i u} \ge z, \quad i = 1, \dots, k-1, k+1, \dots, n$: Since $v_i = w_i - c_i$, $i = 1, \dots, k-1, k+1, \dots, n$, we have

$$zv_i \le v_i = w_i - c_i \le w_i - c_i + (1-z)\phi_i u,$$

from which the inequality follows.
5. $w_i \ge c_i, \quad i = 1, \dots, n$: From our contract design in (21), we find that $w_k \ge c_k$. For $i = 1, \dots, k-1, k+1, \dots, n$, we should show that $v_i \ge 0$. Since $c_k = \min_{i=1,\dots,n} c_i$, we have

$$\begin{aligned}v_i &= \frac{1-z}{cz} uc_i + \left(\frac{1-z}{cz}u - 1\right)\frac{c_k}{n-1} \\ &\ge \frac{1-z}{cz} uc_k + \left(\frac{1-z}{cz}u - 1\right)\frac{c_k}{n-1} \\ &= \left(\frac{1-z}{cz}u - \frac{1}{n}\right)\frac{nc_k}{n-1} \\ &= \left(\frac{u}{p^* - m + u - c} - \frac{1}{n}\right)\frac{nc_k}{n-1} \\ &\ge \left(\frac{1}{2} - \frac{1}{n}\right)\frac{nc_k}{n-1} \\ &\ge 0.\end{aligned}$$

These arguments show that the properly-designed contracts with the parameters given in (21) and (22) can realize supply chain coordination. $\blacksquare$

Proof of Theorem 6. We first consider the best response decisions of supplier S_i, $i = 1, 2, \dots, n$. We calculate the optimal quantity for supplier S_i for two different cases: (i) $Q_{-i} \le q_i \le y(p) + B$, and (ii) $y(p) + A \le q_i \le Q_{-i}$.

(i) $Q_{-i} \le q_i \le y(p) + B$: In this case, $Q = Q_{-i}$ and S_i's profit function (23) is reduced to

$$\Pi_{S_i} = (w_i - s_i)Q_{-i} + q_i(s_i - c_i).$$

Hence, the expected profit is

$$E(\Pi_{S_i}) = (w_i - s_i)Q_{-i} + q_i(s_i - c_i),$$

which shows that $E(\Pi_{S_i})$ is decreasing in q_i due to $s_i < c_i$. As a result,

$$q_i^* = Q_{-i}, \text{ for } Q_{-i} \leq q_i \leq y(p) + B.$$

(ii) $y(p) + A \leq q_i \leq Q_{-i}$: In this case S_i's profit function (23) becomes $\Pi_{S_i} = (w_i - c_i)q_i$, and the expected profit is

$$E(\Pi_{S_i}) = (w_i - c_i)q_i,$$

which implies that S_i's expected profit $E(\Pi_{S_i})$ is increasing in q_i because $w_i \geq c_i$. Thus,

$$q_i^* = Q_{-i}, \text{ for } y(p) + A \leq q_i \leq Q_{-i}.$$

From our above analysis we conclude that all suppliers' optimal quantities must be equal. As a result, we can write supplier S_i's expected profit as

$$E(\Pi_{S_i}) = (w_i - c_i)q_i,$$

which is increasing in q_i. Because $q_i \leq y(p) + B$, the supplier's best-response quantity is

$$q_i^B = y(p) + B.$$

Next, we compute the manufacturer's best-response retail price p^B. Using Theorem 2 we can uniquely determine the manufacturer's best response price p^B by solving the function

$$\int_A^{Q-y(p)} F(x)dx + (p - m + u)bF(Q - y(p)) = Q + ub.$$

Since $Q = y(p) + B$, we can re-write the above equation to

$$\int_A^B F(x)dx + (p - m)b = y(p) + B,$$

and we can solve the equation to find

$$p^B = \frac{a + mb + B}{2b} - \frac{1}{2b}\int_A^B F(x)dx.$$

Using the above analysis we can find the Nash equilibrium

$$\begin{aligned} p^N &= \frac{a + mb + B}{2b} - \frac{1}{2b}\int_A^B F(x)dx, \\ q_i^N &= \frac{a - mb + B}{2} + \frac{1}{2}\int_A^B F(x)dx, \end{aligned}$$

for $i = 1, \ldots, n$ and $q^N \equiv q_1^N = \cdots = q_n^N$.

Note that $p^N > 0$ and $q^N > 0$ because $\int_A^B F(x)dx < B-A < B$ and $a+B > a+A > pb > mb$. This theorem thus proves. ■

Proof of Theorem 7. This theorem follows from Remark 1, which indicates that the supply chain cannot be coordinated if the buy-back and the lost-sales cost-sharing contracts are not included. ■

Proof of Theorem 8. Shapley value, developed by Shapley [16], is a solution concept for cooperative games, which provides a unique scheme for allocating the profit surplus γ among $(n+1)$ players including n suppliers and the manufacturer. The unique Shapley values $\boldsymbol{\gamma} = (\gamma_{S_1}, \ldots, \gamma_{S_n}; \gamma_M)$ are determined by

$$\gamma_j = \frac{\sum_{j \in T}(|T|-1)![n+1-|T|]![v(T)-v(T-j)]}{(n+1)!}, \text{ for } j = S_1, \ldots, S_n, M.$$

where T denotes a coalition in which some supply chain members cooperate, $v(\cdot)$ denote the profit surplus joint achieved by all cooperative members in a coalition, and $|T|$ is the size of T.

Because, in the assembly supply chain, all suppliers sell *complementary* components to the manufacturer who assembles and sells final products, the system-wide profit surplus would be zero if not all members in such a supply chain cooperate. This means that, if and only if n suppliers and the manufacturer cooperate, then the system-wide profit surplus will be γ. Therefore, we find that, for $j = S_1, \ldots, S_n, M$,

$$\begin{aligned}
\gamma_j &= \frac{\sum_{j \in T}(|T|-1)![(n+1)-|T|]![v(T)-v(T-j)]}{(n+1)!} \\
&= \frac{[(n+1)-1]! \times 1! \times (\gamma - 0)}{(n+1)!} \\
&= \frac{\gamma}{n+1}.
\end{aligned}$$

As a result, all supply chain members' profits after receiving γ_j $(j = j = S_1, \ldots, S_n, M)$ can be computed as shown in this theorem. ■

Proof of Theorem 9. We need to show that the values of parameters specified in this theorem satisfy (30) and (20). It is easy to see that conditions in (30) are satisfied when $u \le c$ or $u > c$. Now we examine whether the values of the parameters can satisfy (20).

1. If $u \le c$, then we find
   (a) $w_i - c_i \le v_i \le w_i, \quad i = 1, \ldots n$: Since $w_i = c_i$, $0 < z < 1$ and $u \le c$, we can easily find that $0 \le v_i \le w_i$, for $i = 1, 2, \ldots, n$.
   (b) $0 \le \phi_i \le 1, \quad i = 1, \ldots, n$: It is easy to show that $0 \le \phi_i \le 1$ $(i = 1, 2, \ldots, n)$ because $z < 1$ and $c_i \le c$.
   (c) $\sum_{i=1}^n \phi_i \le 1$: In such a contract design, we have
   $$\sum_{i=1}^n \phi_i = z < 1.$$
   (d) $\dfrac{w_i - c_i + \phi_i u}{v_i + \phi_i u} \ge z, \quad i = 1, \ldots, k-1, k+1, \ldots, n$: Since $w_i = c_i$, we find that

$(w_i - c_i + \phi_i u)/(v_i + \phi_i u) = z$.

(e) $w_i \geq c_i, \quad i = 1, \ldots, n$: In the contract design, we set $w_i = c_i$, for $i = 1, 2, \ldots, n$.

2. If $u > c$, then we find

(a) $w_i - c_i \leq v_i \leq w_i, \quad i = 1, \ldots n$: Since $w_i = c_i$ and $0 < z < 1$, we can easily find that $0 \leq v_i = (1-z)c_i \leq c_i = w_i$, for $i = 1, 2, \ldots, n$.

(b) $0 \leq \phi_i \leq 1, \quad i = 1, \ldots, n$: It is easy to show that $0 \leq \phi_i \leq 1$ $(i = 1, 2, \ldots, n)$ because $z < 1$ and $c_i \leq c < u$.

(c) $\sum_{i=1}^n \phi_i \leq 1$: In such a contract design, we have

$$\sum_{i=1}^{n} \phi_i = z\frac{c}{u} < 1,$$

because $0 < z < 1$ and $u > c$.

(d) $\dfrac{w_i - c_i + \phi_i u}{v_i + \phi_i u} \geq z, \quad i = 1, \ldots, k-1, k+1, \ldots, n$: Since $w_i = c_i$, we find that $(w_i - c_i + \phi_i u)/(v_i + \phi_i u) = z$.

(e) $w_i \geq c_i, \quad i = 1, \ldots, n$: In the contract design, we set $w_i = c_i$, for $i = 1, 2, \ldots, n$.

These arguments show that the properly-designed contracts with the parameters given in this theorem can realize supply chain coordination. ■

Proof of Theorem 10. From the proof of Theorem 6 we find the best-response decision of supplier S_i $(i = 1, 2, \ldots, n)$ as

$$q_i^B = y(p) + B.$$

To find the Stackelberg equilibrium, we should substitute q_i^B into the manufacturer's profit function and then maximize the resulting expected profit for the optimal price. Using (6) we write the manufacturer's expected profit without the contractual mechanism as

$$E(\Pi_M) = (p-m)[y(p)+\mu] - wQ + (p-m+u) \times \int_{Q-y(p)}^{B} [Q - y(p) - x] f(x)dx.$$

Since $Q = q^B = y(p) + B$, we re-write the manufacturer's expected profit as

$$\begin{aligned} E(\Pi_M) &= (p-m)[y(p)+\mu] - w[y(p)+B] \\ &= (p-m-w)y(p) + (p-m)\mu - wB. \end{aligned}$$

Differentiating $E(\Pi_M)$ w.r.t. p once and twice gives

$$\begin{aligned} \frac{\partial E(\Pi_M)}{\partial p} &= -2bp + a + b(m+w) + \mu, \\ \frac{\partial^2 E(\Pi_M)}{\partial p^2} &= -2b < 0. \end{aligned}$$

Therefore, we can solve $\partial E(\Pi_M)/\partial p = 0$ for p and find

$$p^S = [a + b(m+w) + \mu]/2b, \tag{50}$$

and thus,

$$q_i^S = y(p^S) + B = [a - b(m+w) - \mu]/2 + B, \tag{51}$$

for $i = 1, \ldots, n$ and $q_1^S = \cdots = q_n^S$.

Next, we show that we find that the supply chain cannot be coordinated if the buy-back and lost-sales cost-sharing contracts are not involved. By comparing (28) and (29) with (50) and (51), we find that, for any value of w_i ($i = 1, 2, \ldots, n$), the Stackelberg equilibrium cannot be identical the globally-optimal solutions given in Theorem 4. This theorem thus proves. ∎

Proof of Theorem 11. Similar to Section 2, the production quantities chosen by all suppliers are equal, i.e., $q_1 = q_2 = \ldots = q_n$. Using (31) we develop the manufacturer's and n suppliers' expected profit function as follows:

$$\begin{aligned} E(\Pi_M) &= (p - m - v)y(p)\mu + (v - w)Q \\ &\quad + [p - m + (1-\phi)u - v] \int_{Q/y(p)}^{B} [Q - y(p)x] f(x) dx, \qquad (52) \\ E(\Pi_{S_i}) &= (w_i - c_i)q_i - v_i \int_A^{q_i/y(p)} [q_i - y(p)x] f(x) dx - \phi_i u \int_{q_i/y(p)}^{B} [y(p)x - q_i] f(x) dx. \qquad (53) \end{aligned}$$

We first find the Nash equilibrium for the simultaneous-move game. The first- and second-order derivatives of $E(\Pi_M)$ in (52) w.r.t. p are

$$\begin{aligned} \frac{dE(\Pi_M)}{dp} &= Q - QF\left(\frac{Q}{y(p)}\right) + \left[1 - (p - m - v)\frac{b}{p}\right] y(p)\mu \\ &\quad - \left\{1 - [p - m + (1-\phi)u - v]\frac{b}{p}\right\} \int_{Q/y(p)}^{B} y(p)xf(x)\ dx. \qquad (54) \end{aligned}$$

$$\begin{aligned} \frac{d^2E(\Pi_M)}{dp^2} &= -Qf\left(\frac{Q}{y(p)}\right) \frac{b}{p}\frac{Q}{y(p)} - 2\frac{b}{p}y(p)\mu + (p - m - v)\frac{b(1+b)}{p^2}y(p)\mu \\ &\quad + \frac{b}{p}\int_{Q/y(p)}^{B} y(p)xf(x)\ dx - [p - m + (1-\phi)u - v]\frac{b}{p^2}\int_{Q/y(p)}^{B} y(p)xf(x)\ dx \\ &\quad + \left\{1 - [p - m + (1-\phi)u - v]\frac{b}{p}\right\} Qf\left(\frac{Q}{y(p)}\right) \\ &\quad + \left\{1 - [p - m + (1-\phi)u - v]\frac{b}{p}\right\} \frac{b}{p}\int_{Q/y(p)}^{B} y(p)xf(x)\ dx \qquad (55) \end{aligned}$$

Since the function (55) is very complicated, we cannot determine the sign of $d^2E(\Pi_M)/dp^2$ and the concavity of the manufacturer's profit function $E(\Pi_M)$. However, as Petruzzi and Dada [14, Theorem 2] have argued, a single-period profit function with the price-dependent demand either in the multiplicative or additive form need not be concave or unimodal, but a finite optimal price always exists. Thus, we conclude that a finite optimal price that maximizes $E(\Pi_M)$ exists and it must satisfy the condition that $dE(\Pi_M)/dp = 0$.

Thus, for the multiplicative case, the manufacturer's best-response retail price p^B, for the given contract parameters $(\mathbf{w}, \mathbf{v}; \boldsymbol{\phi})$ and suppliers' quantity decisions $\mathbf{q} = (q_1, \ldots, q_n)$, satisfies the following condition:

$$\begin{aligned} & Q+\left[1-(p-m-v)\frac{b}{p}\right]y(p)\mu \\ = \;& QF\left(\frac{Q}{y(p)}\right)+\left\{1-[p-m+(1-\phi)u-v]\frac{b}{p}\right\}\int_{Q/y(p)}^{B} y(p)xf(x)dx. \end{aligned} \tag{56}$$

Similar to Section 2, all suppliers choose their production quantities equal to the production quantity of the supplier S_k, who has the minimum value for the right-hand-side term in (4). Thus, we should only analyze supplier S_k's production decision. Replacing the subscript "i" in (53) with "k", we can determine the supplier S_k's expected profit function $E(\Pi_{S_k})$. The first- and second-order derivatives of S_k's expected profit function w.r.t. q_i are

$$\begin{aligned} \frac{dE(\Pi_{S_k})}{dq_k} &= w_k-c_k+\phi_k u-(v_k+\phi_k u)F\left(\frac{q_k}{y(p)}\right), \\ \frac{d^2E(\Pi_{S_k})}{dq_k^2} &= (v_k+\phi_k u)f\left(\frac{q_k}{y(p)}\right)\frac{q_k}{[y(p)]^2}y'(p)<0, \end{aligned}$$

which implies that S_k's expected profit $E(\Pi_{S_k})$ is a strictly concave function of q_k. Thus, we can solve $dE(\Pi_{S_k})/dq_k=0$ and obtain the supplier's best-response production quantity that maximizes $E(\Pi_{S_k})$ as

$$q_k^B=y(p)F^{-1}\left(\frac{w_k-c_k+\phi_k u}{v_k+\phi_k u}\right)=y(p)F^{-1}\left(z_k\right).$$

Recalling that all suppliers' production quantities are equal to q_k^B, we find that, for a fixed manufacturer's price p, all suppliers' best-response production quantities are

$$q_1^B=q_2^B=\ldots=q_n^B=y(p)F^{-1}\left(z_k\right), \tag{57}$$

where $z_k=(w_k-c_k+\phi_k u)/(v_k+\phi_k u)$ with the index k defined by (4).

By using the best-response functions in (56) and (57), we can now find the Nash equilibrium for the simultaneous-move game, as shown in this theorem.

Next, we find the Stackelberg equilibrium for the leader-follower game. Substituting n suppliers' best-response quantity in (57) into the manufacturer's expected profit function (52), we have

$$\begin{aligned} E(\Pi_M) \;=\;& (p-m-v)y(p)\mu+(v-w)y(p)F^{-1}\left(z_k\right) \\ & +[p-m+(1-\phi)u-v]y(p)\int_{F^{-1}(z_k)}^{B}[F^{-1}\left(z_k\right)-x]f(x)dx. \end{aligned}$$

The first-order derivative of $E(\Pi_M)$ w.r.t. p is

$$\begin{aligned}\frac{\partial E(\Pi_M)}{\partial p} &= y(p)\mu - b(p-m-v)y(p)\mu/p - b(v-w)y(p)F^{-1}(z_k)/p \\ &+y(p)\int_{F^{-1}(z_k)}^{B}[F^{-1}(z_k)-x]f(x)dx \\ &-b[p-m+(1-\phi)u-v]y(p)\int_{F^{-1}(z_k)}^{B}[F^{-1}(z_k)-x]f(x)dx/p\end{aligned}$$

Similar to our analysis for the simultaneous-move game, a finite optimal price that maximizes $E(\Pi_M)$ for the leader-follower game exists and it must satisfy the condition that $dE(\Pi_M)/dp = 0$, which can be re-written as

$$\begin{aligned}&F^{-1}(z_k)[1-z_k] + \left[1-(p^S-m-v)\frac{b}{p^S}\right]\mu \\ =\ &\{[p^S-m+(1-\phi)u-v](1-z_k)+(v-w)\}\frac{b}{p^S}F^{-1}(z_k) \\ &+\left\{1-[p^S-m+(1-\phi)u-v]\frac{b}{p^S}\right\}\int_{F^{-1}(z_k)}^{B}xf(x)dx\end{aligned}$$

Thus, we use (57) to find that $q_1^S = q_2^S = \ldots = q_n^S = y(p^S)F^{-1}(z_k)$. This theorem proves. ■

Proof of Theorem 12. The first-order partial derivatives of $E(\Pi)$ w.r.t. p and q are

$$\frac{\partial E(\Pi)}{\partial p} = q\left[1-F\left(\frac{q}{y(p)}\right)\right] + \left[1-(p-m)\frac{b}{p}\right]y(p)\int_A^{q/y(p)}xf(x)dx + u\frac{b}{p}y(p)\int_{q/y(p)}^{B}xf(x)dx,$$

and

$$\frac{\partial E(\Pi)}{\partial q} = -c + (p-m+u)\left[1-F\left(\frac{q}{y(p)}\right)\right].$$

The second-order partial derivatives are computed as

$$\begin{aligned}\frac{\partial^2 E(\Pi)}{\partial p^2} &= -f\left(\frac{q}{y(p)}\right)\frac{q^2 b}{py(p)} - \frac{mb}{p^2}\int_A^{q/y(p)}y(p)xf(x)dx \\ &+(b-1)u\frac{b}{p^2}\int_{q/y(p)}^{B}y(p)xf(x)dx + \left[1-(p-m+u)\frac{b}{p}\right]qf\left(\frac{q}{y(p)}\right), \\ \frac{\partial^2 E(\Pi)}{\partial q^2} &= -\frac{p-m+u}{y(p)}f\left(\frac{q}{y(p)}\right) < 0.\end{aligned}$$

Thus, for a fixed p, the chainwide profit $E(\Pi)$ is strictly concave in q. However, since $\partial^2 E(\Pi)/\partial p^2$ is too complicated to analyze explicitly, it is difficult to reach a conclusion about the concavity or unimodality of $E(\Pi)$. But, as Petruzzi and Dada [14, Theorem 2] have shown, a unique, finite optimal price p^* always exists for the multiplicative case, and thus, the optimal solution (p^*, q^*) must satisfy the conditions that $\partial E(\Pi)/\partial p = 0$ and $\partial E(\Pi)/\partial q = 0$.

To find the globally optimal solution p^*, we solve $\partial E(\Pi)/\partial q = 0$ for q^* and find (37). Substituting (37) into the condition $\partial E(\Pi)/\partial p = 0$ and simplifying it, we reach the result given by (36). ■

Proof of Theorem 13. We first show that, as b increases, the Nash equilibrium price decreases. We assume that our current Nash equilibrium price is p_1^N. According to the proof of Theorem 11, we re-write the first-order condition for the Nash equilibrium price p_1^N as

$$\begin{aligned}
\left.\frac{\partial E(\Pi)}{\partial p}\right|_{p=p_1^N} &= \frac{1}{b}\left[F^{-1}(z_k)(1-z_k)+\int_A^{F^{-1}(z_k)} xf(x)dx\right] \\
&\quad -(p_1^N-m-v)\frac{\mu}{p_1^N} \\
&\quad +[p_1^N-m+(1-\phi)u-v]\frac{1}{p_1^N}\int_{F^{-1}(z_k)}^{B} xf(x)dx \\
&= 0.
\end{aligned}$$

If the value of b increases, then the term $\frac{1}{b}[F^{-1}(z_k)(1-z_k)+\int_A^{F^{-1}(z_k)} xf(x)dx]$, which is positive, decreases and thus,

$$\left.\frac{\partial E(\Pi)}{\partial p}\right|_{p=p_1^N} < 0,$$

which means that we should reduce the price so as to increase the manufacturer's profit $E(\Pi)$. Thus, the new Nash equilibrium price, denoted by p_2^N, should be less than p_1^N, i.e., $p_2^N < p_1^N$.

We then show that, as b increases, the Stackelberg equilibrium price decreases. We assume that our current Stackelberg equilibrium price is p_1^S. According to the proof of Theorem 11, we re-write the first-order condition for the Stackelberg equilibrium price p_1^S as

$$\begin{aligned}
\left.\frac{\partial E(\Pi)}{\partial p}\right|_{p=p_1^S} &= \frac{1}{b}\left[F^{-1}(z_k)(1-z_k)+\int_A^{F^{-1}(z_k)} xf(x)dx\right] \\
&\quad +(p_1^S-m-v)\frac{\mu}{p_1^S} \\
&\quad -\{[p_1^S-m+(1-\phi)u-v](1-z_k)+(v-w)\}\frac{F^{-1}(z_k)}{p_1^S} \\
&\quad +[p_1^S-m+(1-\phi)u-v]\frac{1}{p_1^S}\int_{F^{-1}(z_k)}^{B} xf(x)dx.
\end{aligned}$$

If the value of b increases, then

$$\left.\frac{\partial E(\Pi)}{\partial p}\right|_{p=p_1^S} < 0,$$

which means that we should reduce the price so as to increase the manufacturer's profit $E(\Pi)$. Thus, the new Stackelberg equilibrium price, denoted by p_2^S, should be less than p_1^S, i.e., $p_2^S < p_1^S$.

Similarly, we can also prove that, as b increases, the globally-optimal solution p^* decreases. ■

Proof of Theorem 14. We first consider supply chain coordination for the simultaneous-move game. In order to coordinate the supply chain, we should equate (32) to (36) and equate (33) to (37), and simplifying them, the conditions that will assure $p^N = p^*$, and $q^N = q^*$ are found as

$$
\begin{aligned}
v \int_{A}^{q^*/y(p^*)} x f(x) dx &= \phi u \int_{q^*/y(p^*)}^{B} x f(x) dx, && (58) \\
\frac{p^* - m + u - c}{p^* - m + u} &= \frac{w_k - c_k + \phi_k u}{v_k + \phi_k u}. && (59)
\end{aligned}
$$

In order to facilitate our discussion, we first define

$$
\lambda \equiv \int_{q^*/y(p^*)}^{B} x f(x) dx,
$$

and since we have denoted the LHS of (59) by z in Section 2.2.2, we can re-write the conditions (58) and (59) as

$$
\left\{ \begin{array}{l} v(\mu - \lambda) = \phi u \lambda, \\ z = \dfrac{w_k - c_k + \phi_k u}{v_k + \phi_k u}. \end{array} \right. \tag{60}
$$

We now show that, for both simultaneous-move and leader-follower games, there always exists a feasible solution satisfying the conditions (20) and (60), thus giving rise to existence of the pair of contracts. However, different from our analysis in Section 2.2.2, we should consider the following two cases: (i) $\kappa \equiv \lambda u/[c(\mu - \lambda)] \leq 1$ and (ii) $\kappa > 1$. For each case, we find proper-designed buy-back and lost-sales cost-sharing contracts, as given in Table 2.

Next, for the simultaneous-move game, we prove that, for each case, the values of parameters specified in Table 2 satisfy (60) and (20). It is easy to see that conditions in (60) are satisfied for both Cases (i) and (ii). Now we investigate whether the values of the parameters in each case can meet the five requirements in (20).

(i) If $\kappa \leq 1$, we use the corresponding values of contract parameters in Table 2 to examine the satisfaction of five requirements (20).

(a) $w_i - c_i \leq v_i \leq w_i, \quad i = 1, \ldots n$: Since $w_k = c_k + z v_k$ and $z = \kappa \leq 1$, we have $v_k \leq c_k$ and $w_k - c_k \leq v_k \leq w_k$. Furthermore, due to $w_i = v_i + c_i$, we have $w_i - c_i \leq v_i \leq w_i$, $i = 1, \ldots, k-1, k+1, \ldots, n$.

(b) $0 \leq \phi_i \leq 1, \quad i = 1, \ldots, n$: From Table 2, we have $\phi_k = 0$. Since $c = \sum_{j=1}^{n} c_j > c_i + c_k > c_i + c_k/(n-1)$, we find that $0 \leq \phi_i \leq 1$, $i = 1, \ldots, k-1, k+1, \ldots, n$.

(c) $\sum_{i=1}^{n} \phi_i \leq 1$: In such a contract design, we have

$$
\begin{aligned}
\sum_{i=1}^{n} \phi_i &= \phi_1 + \cdots + \phi_{k-1} + \phi_k + \phi_{k+1} + \cdots + \phi_n \\
&= \frac{(c_1 + \cdots + c_{k-1} + c_{k+1} + \cdots + c_n) + c_k}{c} + \phi_k \\
&= 1.
\end{aligned}
$$

(d) $\dfrac{w_i - c_i + \phi_i u}{v_i + \phi_i u} \geq z, \quad i = 1, \ldots, k-1, k+1, \ldots, n$: Since $v_i = w_i - c_i$, $i = 1, \ldots, k-1, k+1, \ldots, n$, we have

$$
z v_i \leq v_i = w_i - c_i \leq w_i - c_i + (1 - z) \phi_i u,
$$

from which the inequality follows.

(e) $w_i \geq c_i, \quad i = 1, \ldots, n$: From Table 2, we can easily find that $w_i \geq c_i$, for $i = 1, \ldots n$.

(ii) If $\kappa \geq 1$, we use the contract design in Table 2 to show that all requirements in (20) can be satisfied.

(a) $w_i - c_i \leq v_i \leq w_i, \quad i = 1, \ldots n$: Since $zc_k \leq c_k \leq (1+z)c_k$, we have $w_k - c_k \leq v_k \leq w_k$. Furthermore, due to $w_i = v_i + c_i$, we have $w_i - c_i \leq v_i \leq w_i$, $i = 1, \ldots, k-1, k+1, \ldots, n$.

(b) $0 \leq \phi_i \leq 1, \quad i = 1, \ldots, n$: The proof for this requirement is the same as that in Case (1).

(c) $\sum_{i=1}^{n} \phi_i \leq 1$: The proof for this requirement is the same as that in Case (1).

(d) $\dfrac{w_i - c_i + \phi_i u}{v_i + \phi_i u} \geq z, \quad i = 1, \ldots, k-1, k+1, \ldots, n$: The proof for this requirement is the same as that in Case (1).

(e) $w_i \geq c_i, \quad i = 1, \ldots, n$: From our contract design Table 2, we find that $w_k \geq c_k$. For $i = 1, \ldots, k-1, k+1, \ldots, n$, we should show that $v_i \geq 0$. As Table 2 indicates,

$$v_i = \kappa c_i + (\kappa - 1)\frac{c_k}{n-1}$$

Since $\kappa \geq 1$, we have $v_i \geq 0$, which implies that $w_i \geq c_i$, $i = 1, \ldots, k-1, k+1, \ldots, n$.

These arguments show that there exists the properly-designed contracts that can realize supply chain coordination for the simultaneous-move game.

Next, we show that, for the leader-follower game, the properly-designed contracts as given in Table 2 can also coordinate the supply chain. According to Theorem 11 we find that the conditions that will assure $p^S = p^*$, and $q^S = q^*$ are found as,

$$\begin{cases} v(\mu - \lambda) = \phi u \lambda, \\ z = \dfrac{w_k - c_k + \phi_k u}{v_k + \phi_k u}, \end{cases}$$

where

$$\lambda \equiv \int_{q^*/y(p^*)}^{B} [x - q^*/y(p^*)] f(x) dx.$$

Using our arguments for the simultaneous-move game, we can arrive to the theorem. ■

Proof of Theorem 15. We first solve the simultaneous-move game to find the Nash equilibrium. From the proof of Theorem 11, we find the manufacturer's and supplier S_i's expected profit functions as

$$\begin{aligned} E(\Pi_M) &= (p - m - v)y(p)\mu + (v - w)Q + [p - m + (1-\phi)u - v] \int_{Q/y(p)}^{B} [Q - y(p)x] f(x) dx, \\ E(\Pi_{S_i}) &= (w_i - c_i)q_i - v_i \int_{A}^{q_i/y(p)} [q_i - y(p)x] f(x) dx - \phi_i u \int_{q_i/y(p)}^{B} [y(p)x - q_i] f(x) dx. \end{aligned}$$

When the buy-back and lost-sales cost-sharing contracts are not involved, $v_i = \phi_i = 0$ and thus, we can reduce the above functions to

$$
\begin{aligned}
E(\Pi_M) &= (p-m)y(p)\mu - wQ + (p-m+u)\int_{Q/y(p)}^{B} [Q-y(p)x]f(x)dx, \\
E(\Pi_{S_i}) &= (w_i - c_i)q_i.
\end{aligned}
$$

Since $Ay(p) \le q_i \le By(p)$, $i = 1, \ldots, n$, we find that supplier S_i's best-response quantity is $q_i^B = By(p)$.

Next, we first find the Nash equilibrium for the simultaneous-move game. We differentiate $E(\Pi_M)$ w.r.t. p once, and find

$$
\begin{aligned}
\frac{\partial E(\Pi_M)}{\partial p} &= y(p)\mu - b(p-m)y(p)\mu/p \\
&\quad + \int_{Q/y(p)}^{B} [Q-y(p)x]f(x)dx \\
&\quad + by(p)(p-m+u)\int_{Q/y(p)}^{B} xf(x)dx/p.
\end{aligned}
$$

Since $Q = \min(q_1^B, \ldots, q_1^B) = By(p)$, we re-write $\partial E(\Pi_M)/\partial p$ as

$$
\frac{\partial E(\Pi_M)}{\partial p} = y(p)\mu[1 - b(p-m)/p],
$$

and compute $\partial^2 E(\Pi_M)/\partial p^2$ as

$$
\frac{\partial^2 E(\Pi_M)}{\partial p^2} = -\frac{by(p)}{p}\mu[1-b(p-m)/p] - \frac{y(p)\mu bm}{p^2}.
$$

It easily follows that, for any price satisfying $dE(\Pi_M)/dp = 0$, $d^2E(\Pi_M)/dp^2 = -y(p)\mu bm/p^2 < 0$. This implies that the manufacturer's expected profit is a unimodal function of the unit retail price p. Equating $\partial E(\Pi_M)/\partial p$ to zero and solving the resulting equation for p, we find that $p^N = bm/(b-1)$. Thus, $q_i^N = By(p^N)$, $i = 1, \ldots, n$.

Next, we compute the leader-follower game to find the Stackelberg equilibrium. Using the suppliers' best responses, we write the manufacturer's expected profit $E(\Pi_M)$ as

$$
E(\Pi_M) = [(p-m)\mu - wB]y(p).
$$

The first-order derivative of $E(\Pi_M)$ w.r.t. p is

$$
\frac{\partial E(\Pi_M)}{\partial p} = \mu y(p) - b[(p-m)\mu - wB]y(p)/p.
$$

Similar to the above, we can easily prove that $E(\Pi_M)$ is a unimodal function of p. Equating $\partial E(\Pi_M)/\partial p$ to zero and solving the resulting equation for p, we find that

$$
p^S = \frac{b(m\mu + wB)}{(b-1)\mu}.
$$

Thus, $q_i^S = By(p^S)$, $i = 1, \ldots, n$.

Comparing the above Nash and Stackelberg equilibria with the globally-optimal solution in Theorem 12, we find that, for any value of w_i, the supply chain cannot be coordinated. Therefore, if the buy-back and lost-sales cost-sharing contracts are not involved, then, for any value of w_i ($i = 1, 2, \ldots, n$), the total profit in terms of the Nash or Stackelberg equilibrium is always smaller than that in terms of the globally-optimal solutions given in Theorem 12. ■

Appendix B Numerical Examples for the Additive Demand Case

In this appendix, we consider a three-supplier ($n = 3$), one-manufacturer assembly supply chain for both simultaneous-move and leader-follower games, and, for each game, construct the buyback and lost-sales cost-sharing contracts to achieve supply chain coordination. With the proposed contract, all suppliers and the manufacturer choose the equilibrium solution which results in the maximum expected system-wide profit. Then, in order to assure that all supply chain members are better off under the properly-designed contracts, we use (25) to compute the system-wide expected profit surplus γ, and use Theorem 8 to determine the allocation of γ among n suppliers and the manufacturer.

In our examples for both the simultaneous-move and leader-following games, we use the following parameter values:

c_1	c_2	c_3	s_1	s_2	s_3	m	a	b	A	B
2	3	5	1	2	2	2	20	0.8	4	12

We assume that the error term ε in demand function (1) is uniformly distributed with probability density function $f(x) = 1/(B-A)$ and cumulative distribution function $F(x) = (x-A)/(B-A)$ for $A \leq x \leq B$. Next, we first consider an example to illustrate our analysis for the simultaneous-move game.

Example 1 We use the five steps given in Section 2.2.2 to develop the proper contracts for supply chain coordination.

Step 1: To find the globally optimal solution, we use $u = 12$ and find $(p^*, q^*) = (23.27, 11)$ for which the requirement $u > p^* - m - c$ is satisfied since $m = 2$ and $c = 2 + 3 + 5 = 10$.

Step 2: We compute $z = (p^* - m + u - c)/(p^* - m + u) = 0.699$.

Step 3: The critical supplier is S_1 since this supplier's unit production cost c_1 is the smallest; so $k = 1$.

Step 4: We set supplier S_1's wholesale price as $w_1 = (1 + z)c_1 = \$3.40$/unit, buyback price as $v_1 = c_1 = \$2$/unit, and $\phi_1 = 0$;

Step 5: The other suppliers' wholesale prices, buyback prices and percentage of the underage cost absorbed by them are

$$\begin{cases} w_2 = \left(\dfrac{1-z}{cz}u+1\right)c_2 + \left(\dfrac{1-z}{cz}u-1\right)\dfrac{c_1}{n-1} = \$4.06/\text{unit}, \\ v_2 = \dfrac{1-z}{cz}uc_2 + \left(\dfrac{1-z}{cz}u-1\right)\dfrac{c_1}{n-1} = \$1.06/\text{unit}, \\ \phi_2 = \dfrac{1}{c}\left(c_2 + \dfrac{c_1}{n-1}\right) = 0.4, \end{cases}$$

and

$$\begin{cases} w_3 = \left(\dfrac{1-z}{cz}u+1\right)c_3 + \left(\dfrac{1-z}{cz}u-1\right)\dfrac{c_1}{n-1} = \$7.09/\text{unit}, \\ v_3 = \dfrac{1-z}{cz}uc_3 + \left(\dfrac{1-z}{cz}u-1\right)\dfrac{c_1}{n-1} = \$2.09/\text{unit}, \\ \phi_3 = \dfrac{1}{c}\left(c_3 + \dfrac{c_1}{n-1}\right) = 0.6. \end{cases}$$

To summarize, under the buyback and lost-sales cost-sharing contracts, we have

$$\mathbf{w} = (3.40, 4.06, 7.09), \quad \mathbf{v} = (2.00, 1.06, 2.09), \quad \text{and} \quad \boldsymbol{\phi} = (0, 0.4, 0.6),$$

where each supplier produces $q^* = 11$ units of each component, and the manufacturer assembles the final products and sells them to the ultimate customer at the retail price $p^* = \$23.27$/unit. Furthermore, since $u = 12$, the suppliers S_2 and S_3 respectively absorb the underage cost of $u\phi_2 = \$4.8$ and $u\phi_3 = \$7.2$ for each unit of lost-sale at the manufacturer's level. Finally, the expected profits of the manufacturer and all suppliers are computed as

$$E(\Pi_M^*) = \$42.21; \quad E(\Pi_{S_1}^*) = \$11.44, \quad E(\Pi_{S_2}^*) = \$7.85, \quad E(\Pi_{S_3}^*) = \$16.28,$$

and maximum system-wide expected profit is $E(\Pi^*) = \$77.78$.

Next, we use Theorem 6 to find the Nash equilibrium without the buyback and lost-sales cost-sharing contracts as $p^N = 18.5$ and $q_i^N = 17.2$, for $i = 1, 2, 3$. In order to calculate all supply chain members' expected profits, we must determine supplier S_i's wholesale price w_i. Note that, for n suppliers, $w \geq c = \sum_{i=1}^{3} c_i = 10$ in order to assure that these suppliers' total profit is non-negative. Moreover, $w \leq (p^N - m)[y(p^N) + \mu]/[y(p^N) + B] = 12.66$, in order to assure that the manufacturer's profit is non-negative. Hence, when the buy-back and lost-sales cost-sharing contracts are not involved, the manufacturer and n suppliers should choose a proper value of w between 10 and 12.66, i.e., $10 \leq w \leq 12.66$.

We note that, if the value of w increases, then n suppliers' aggregate profit (i.e., $w - c$) increases but the manufacturer's profit (i.e., $p - w$) decreases. On the other hand, if the value of w decreases, then n suppliers' aggregate profit decreases but the manufacturer's profit increases. We assume that, for the supply chain without the buyback and lost-sales cost-sharing contracts, the manufacturer and n suppliers have equal bargaining powers to determine the value of w, and thus we set $w = (10 + 12.66)/2 = 11.33$ because $10 \leq w \leq 12.66$. We also note that $\sum_{i=1}^{3} w_i = w$, $c_i \leq w_i \leq w$ $(i = 1, 2, 3)$, and supplier S_i's wholesale price w_i should be dependent of its production cost c_i. That is, if S_i's production cost c_i is higher than supplier S_j's production cost c_j $(j = 1, 2, 3,\ j \neq i)$, then S_i's wholesale price w_i should be also higher than S_j's wholesale price w_j; otherwise, if $c_i < c_j$, then $w_i < w_j$. To reflect this, we assume that, for this example, S_i's wholesale price w_i is proportional to its production cost c_i. Since $\sum_{i=1}^{n} w_i = w$ and $\sum_{i=1}^{3} c_i = c$, we set $w_i = w \times c_i/c$, for $i = 1, 2, \ldots, n$. We use the above method to calculate n suppliers' wholesale prices for all subsequent examples.

Using the above, we calculate the expected profits as

$$E(\tilde{\Pi}_M) = \$22.9; \quad E(\tilde{\Pi}_{S_1}) = \$4.58, \quad E(\tilde{\Pi}_{S_2}) = \$6.87, \quad E(\tilde{\Pi}_{S_3}) = \$11.45,$$

and total system-wide expected profit is $E(\tilde{\Pi}) = \$45.8$, which is smaller than $E(\Pi^*)$, as shown in Theorem 7. Even though the manufacturer and three suppliers all benefit from the proper contracts for supply chain coordination, we notice that their individual profit surpluses are different; for example, the manufacturer's and supplier S_1's individual surpluses are $\$42.21 - \$22.9 = \$19.31$ and $\$11.44 - \$4.58 = \$6.86$, respectively. This may discourage the member with smaller individual surplus from cooperating for supply chain coordination. To address the problem, we use (25) to compute the system-wide profit surplus as $\gamma = E(\Pi^*) - E(\tilde{\Pi}) = \$31.98$, and use Theorem 8 to determine the allocation as $\gamma_M = \gamma_{S_1} = \gamma_{S_2} = \gamma_{S_3} = \gamma/4 = \$7.995$. As a result, after allocating γ, the manufacturer's and three suppliers' profits are $E(\tilde{\Pi}_M) + \gamma_M = \$30.895$; $E(\tilde{\Pi}_{S_1}) + \gamma_{S_1} = \$12.575$, $E(\tilde{\Pi}_{S_2}) + \gamma_{S_2} = \$14.865$, and $E(\tilde{\Pi}_{S_3}) + \gamma_{S_3} = \$19.445$. $\diamondsuit$

In Example 1 we have obtained the properly-designed buy-back and lost sales cost-sharing contracts to induce supply chain coordination, under which Nash equilibrium (p^N, q^N) is identical to the globally optimal solution (p^*, q^*). In order to demonstrate that our contract design in Example 1 maximizes system-wide expected profit and coordinates the supply chain, we deviate from the optimal design, and compute (i) the resulting Nash equilibrium, (ii) expected profits of the manufacturer and three suppliers and (iii) expected system-wide profit. We then compare the expected system-wide profit resulting from the Nash equilibrium with the results obtained under our properly designed contract.

We consider a total of 21 cases where one or more contract parameters are increased or decreased and other parameters are fixed as shown in Table 3. In particular, we first set $w_1 = 4$ (which is above the contract design value of 3.40) while keeping the other parameter values fixed and compute the Nash equilibrium values (p^N, q^N), and the resulting expected profits for the suppliers, the manufacturer and the system-wide expected profit. Since the new system-wide expected profit will be lower than that obtained under the contract design, we also note the percentage reduction in the expected profit when each parameter assumes a value different from the one used in the contract design. For this case we note that the system-wide expected profit is approximately 16% lower than that obtained under the contract design. We repeat the same calculations with $w_1 = 3$ (which is below the contract design value of 3.40) and calculate the resulting expected profits. Next, we set $v_1 = 2.5$ (above the contract value of 2.0) and $v_1 = 1.5$ (below the contract value of 2.0) and repeat the calculations. Similar calculations are performed by assigning different values to pairs of design parameters (w_2, v_2), (w_3, v_3), (ϕ_1, ϕ_2), (ϕ_1, ϕ_3), and (ϕ_2, ϕ_3) while keeping the values of the other parameters fixed at their contract design levels. We complete the calculations by assigning five different values to the triple (ϕ_1, ϕ_2, ϕ_3). When we increase and decrease the value of w_i or v_i, we consider the following constraints:

$w_i - c_i \le v_i \le w_i$ and $w_i \ge c_i$, $i = 1, 2, 3$. Similarly, when we choose new values of the parameters (ϕ_1, ϕ_2, ϕ_3), we make sure that $\phi_1 + \phi_2 + \phi_3 \le 1$.

The results in Table 3 clearly demonstrate that if the parameters deviate substantially from those found in our contract design, the expected system-wide profit may be reduced by large amounts. For example, when $w_1 = 4$ (rather than 3.40 as suggested by our contract design), we see a 16% reduction in the expected system-wide profit. Similarly, when $\mathbf{w} = (0.7, 0.2, 0.1)$ [as opposed to $(0, 0.4, 0.6)$], the reduction is 10%, still a large amount. On the other hand, small deviations from the design parameters result in equally small reductions in system-wide expected profit as in, e.g., $(w_3, v_3) = (7.05, 2.05)$ [as opposed to (7.09,2.09)], and $\boldsymbol{\phi} = (0, 0.6, 0.4)$ [as opposed to $(0, 0.4, 0.6)$], both resulting in a 1% reduction.

Next, we consider another example to illustrate our analysis for the leader-follower game.

Example 2 For the leader-follower game we still consider the parameter values in Example 1. To find properly-designed contracts, we again follow the five steps given in Section 2.2.2 but use Theorem 9 for our contract design, which is given as follows:

$$\mathbf{w} = (2, 3, 5),\ \mathbf{v} = (0.602, 0.903, 1.505),\ \text{and}\ \boldsymbol{\phi} = (0.1165, 0.17475, 0.29125).$$

Under the contracts, the Stackelberg solution is identical to the globally-optimal solution (i.e., $p^S = p^* = \$23.27$ and $q^S = q^* = 11$), and all supply chain members' profits are found as

$$E(\Pi_M^*) = \$86.18; \quad E(\Pi_{S_1}^*) = -\$1.68, \quad E(\Pi_{S_2}^*) = -\$2.52, \quad E(\Pi_{S_3}^*) = -\$4.2,$$

and maximum system-wide expected profit is $E(\Pi^*) = \$77.78$. Note that three suppliers obtain negative profits because, as discussed previously, we must set their wholesale prices w_i equal to their product cost c_i in order to achieve supply chain coordination. Under the contracts, all suppliers' sale profits are zero and they have to buy unused components and share shortage costs; thus, three suppliers' expected profits are all negative. However, the system-wide profits are maximized.

Next, we use Theorem 10 to compute the Stackelberg equilibrium without the buyback and lost-sales cost-sharing contracts as $p^S = 24.83$ and $q_i^S = 12.14$, for $i = 1, 2, \ldots, n$, and the resulting expected profits as

$$E(\tilde{\Pi}_M) = \$32.16; \quad E(\tilde{\Pi}_{S_1}) = 6.44, \quad E(\tilde{\Pi}_{S_2}) = 9.66, \quad E(\tilde{\Pi}_{S_3}) = \$16.1,$$

and total system-wide expected profit is $E(\tilde{\Pi}) = \$64.36$, which is smaller than $E(\Pi^*)$, as shown in Theorem 10. However, we notice that the manufacturer benefits from the proper contracts but three suppliers are worse off than without the contracts. In order to entice three suppliers to cooperate for supply chain coordination, we use (25) to compute the system-wide profit surplus as $\gamma = E(\Pi^*) - E(\tilde{\Pi}) = \$13.42$, and use Theorem 8 to determine the allocation as $\gamma_M = \gamma_{S_1} = \gamma_{S_2} = \gamma_{S_3} = \gamma/4 = \$3.355$. As a result, the manufacturer's and three suppliers' profits are $E(\tilde{\Pi}_M) + \gamma_M = \$35.515$; $E(\tilde{\Pi}_{S_1}) + \gamma_{S_1} = \$9.795$, $E(\tilde{\Pi}_{S_2}) + \gamma_{S_2} = \$13.015$ and $E(\tilde{\Pi}_{S_3}) + \gamma_{S_3} = \$19.455$. $\Diamond$

Parameter(s)	w_1		v_1		(w_2, v_2)		
Changed values	4	3	2.5	1.5	$(9, 6)$	$(4.05, 1.05)$	$(4, 3.8)$
(p^N, q^N)	$(22.95, 13.64)$	$(25.3, 7.76)$	$(25.12, 8.38)$	$(22.96, 13.09)$	$(26.23, 8.61)$	$(24.19, 10.24)$	$(25.42, 9.06)$
$E(\Pi_M)$	15.91	52.85	49.96	27.24	13.19	45.04	52.66
$E(\Pi_{S_1})$	19.28	5.76	8.58	13.09	8.12	10.40	9.02
$E(\Pi_{S_2})$	10.22	2.37	3.83	10.10	38.18	6.96	0.11
$E(\Pi_{S_3})$	20.15	6.93	9.29	19.96	11.29	14.70	12.07
$E(\Pi)$ [%↓]	65.56 [16%]	67.91 [13%]	71.66 [8%]	70.39 [10%]	70.78 [9%]	77.10 [1%]	73.86 [5%]

Parameters	(w_3, v_3)			(ϕ_1, ϕ_2)		(ϕ_1, ϕ_3)	
Changed values	$(14, 9)$	$(7.05, 2.05)$	$(7, 6.5)$	$(0.2, 0.2)$	$(0.4, 0)$	$(0.3, 0.3)$	$(0.5, 0.1)$
(p^N, q^N)	$(27.04, 7.96)$	$(24.19, 10.24)$	$(26.11, 8.49)$	$(23.47, 12.13)$	$(23.28, 12.67)$	$(23.35, 12.46)$	$(23.23, 12.82)$
$E(\Pi_M)$	3.15	45.19	55.65	35.31	31.66	33.14	30.60
$E(\Pi_{S_1})$	7.22	10.41	8.26	10.82	10.92	10.87	10.94
$E(\Pi_{S_2})$	4.62	7.04	5.01	9.52	9.91	9.61	9.85
$E(\Pi_{S_3})$	51.43	14.46	2.14	18.58	19.31	19.22	19.61
$E(\Pi)$ [%↓]	66.42 [15%]	77.11 [1%]	71.06 [9%]	74.23 [5%]	71.80 [8%]	72.84 [6%]	71.00 [9%]

Parameters	(ϕ_2, ϕ_3)		(ϕ_1, ϕ_2, ϕ_3)				
Changed values	$(0.3, 0.7)$	$(0.6, 0.4)$	$(0.5, 0.3, 0.2)$	$(0.3, 0.5, 0.2)$	$(0.6, 0.3, 0)$	$(0.7, 0.2, 0)$	$(0.7, 0.2, 0.1)$
(p^N, q^N)	$(24.20, 10.23)$	$(24.20, 10.23)$	$(23.23, 12.82)$	$(23.35, 12.46)$	$(23.23, 12.89)$	$(23.20, 12.98)$	$(23.16, 13.01)$
$E(\Pi_M)$	44.97	44.97	30.60	33.14	29.95	29.31	29.17
$E(\Pi_{S_1})$	10.40	10.40	10.94	10.88	10.91	10.93	10.97
$E(\Pi_{S_2})$	7.47	6.17	9.88	9.55	9.90	9.96	9.99
$E(\Pi_{S_3})$	14.26	15.56	19.59	19.27	19.65	19.71	19.75
$E(\Pi)$ [%↓]	77.10 [1%]	77.10 [1%]	71.01 [9%]	72.84 [6%]	70.41 [10%]	69.91 [10%]	69.89 [10%]

Table 3: Nash equilibrium and expected profits when the buy-back and lost-sales cost-sharing contracts differ from the contract design values $\mathbf{w} = (3.40, 4.06, 7.09)$, $\mathbf{v} = (2.00, 1.06, 2.09)$, and $\boldsymbol{\phi} = (0, 0.4, 0.6)$ with $E(\Pi) = 77.78$. The numbers in the brackets $[\cdot]$ indicate the percentage reduction in system-wide expected profit when the parameters deviate from the contract design values.

Appendix C Numerical Examples for the Multiplicative Demand Case

Similar to Section 2 we present two examples—one for the simultaneous-move game and the other for the leader-follower game.

Example 3 For the simultaneous-move game, we now consider a numerical example with the same values of the parameters $(c_1, c_2, c_3, s_1, s_2, s_3)$ and $n = 3$ suppliers used for the additive case in online Appendix B. We assume that $m = 10$, $(a, b) = (1000, 1.5)$ in (31), where $y(p) = ap^{-b}$; and the term ε in (31) is a uniformly-distributed random variable with p.d.f. $f(x) = 1/(B - A)$ and c.d.f. $F(x) = (x - A)/(B - A)$, for $A \leq x \leq B$. For this numerical example, we set $A = 4$ and $B = 8$. Using Theorem 14, the proper contracts for supply chain coordination are found as

$$\mathbf{w} = (2.47, 3.82, 6.36), \quad \mathbf{v} = (0.54, 0.82, 1.36), \quad \text{and} \quad \boldsymbol{\phi} = (0, 0.4, 0.6),$$

We then find the globally optimal solution $(p^*, q^*) = (67.63, 13.35)$ and compute the expected profits of the manufacturer and three suppliers as

$$E(\Pi_M) = \$455.83; \quad E(\Pi_{S_1}) = \$4.79, \quad E(\Pi_{S_2}) = \$8.39, \quad E(\Pi_{S_3}) = \$14.04,$$

and maximum chainwide expected profit is $E(\Pi) = \$483.05$.

Next, we use Theorem 15 to find the Nash equilibrium without the buyback and lost-sales cost-sharing contracts as $p^N = 30$ and $q_i^N = 48.69$, for $i = 1, 2, \ldots, n$, and calculate the resulting expected profits as

$$E(\tilde{\Pi}_M) = \$121.72; \quad E(\tilde{\Pi}_{S_1}) = \$24.34, \quad E(\tilde{\Pi}_{S_2}) = \$36.51, \quad E(\tilde{\Pi}_{S_3}) = \$60.86,$$

and total system-wide expected profit is $E(\tilde{\Pi}) = \$243.43$, which is smaller than $E(\Pi^*)$, as shown in Theorem 15. In addition, we find that, under the contracts, the manufacturer are better off but three suppliers are worse off. We then use (25) to compute $\gamma = 239.62$, and use Theorem 8 to calculate the allocations $\gamma_M = \gamma_{S_1} = \gamma_{S_2} = \gamma_{S_3} = \$59.905$. As a result, the manufacturer's and three suppliers' profits are $E(\tilde{\Pi}_M) + \gamma_M = \$181.625$; $E(\tilde{\Pi}_{S_1}) + \gamma_{S_1} = \$84.245$, $E(\tilde{\Pi}_{S_2}) + \gamma_{S_2} = \$96.415$ and $E(\tilde{\Pi}_{S_3}) + \gamma_{S_3} = \$120.765$. $\diamondsuit$

Next, we provide an example to analyze the leader-follower game.

Example 4 We still use the values of parameters in Example 3, and find the proper contracts as,

$$\mathbf{w} = (2.01, 3.02, 5.04), \quad \mathbf{v} = (0.02, 0.03, 0.05), \quad \text{and} \quad \boldsymbol{\phi} = (0, 0.4, 0.6),$$

compute the expected profits of the manufacturer and three suppliers in the leader-follower game as

$$E(\Pi_M) = \$483.07; \quad E(\Pi_{S_1}) = \$0.15, \quad E(\Pi_{S_2}) = -\$0.09, \quad E(\Pi_{S_3}) = -\$0.1,$$

and maximum chainwide expected profit is $E(\Pi) = \$483.03$. We then calculate the Stackelberg equilibrium when the contracts are not involved as $p^S = 90$ and $q_i^S = 9.37$, for $i = 1, 2, \ldots, n$, and calculate the resulting expected profits as

$$E(\tilde{\Pi}_M) = \$421.63; \quad E(\tilde{\Pi}_{S_1}) = \$9.37, \quad E(\tilde{\Pi}_{S_2}) = \$14.05, \quad E(\tilde{\Pi}_{S_3}) = \$23.42,$$

and total system-wide expected profit is $E(\tilde{\Pi}) = \$468.47$, which is smaller than $E(\Pi^*)$, as shown in Theorem 15. To assure supply chain coordination, we use (25) to compute $\gamma = 14.56$, and use Theorem 8 to calculate the allocations $\gamma_M = \gamma_{S_1} = \gamma_{S_2} = \gamma_{S_3} = \$3.64$. As a result, the manufacturer's and three suppliers' profits are $E(\tilde{\Pi}_M) + \gamma_M = \$425.27$; $E(\tilde{\Pi}_{S_1}) + \gamma_{S_1} = \$13.01$, $E(\tilde{\Pi}_{S_2}) + \gamma_{S_2} = \$17.69$ and $E(\tilde{\Pi}_{S_3}) + \gamma_{S_3} = \$27.06$. $\diamondsuit$