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Game-theoretic analysis of an ancient Chinese horse race problem

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Abstract

This paper analyzes a legendary Chinese horse race problem involving the King of Qi and General Tianji which took place more than 2000 years ago. In this problem each player owns three horses of different speed classes and must choose the sequence of horses to compete against each other. Depending on the payoffs received by the players as a result of the horse races, we analyze two groups of constant-sum games. In each group, we consider three separate cases where the outcomes of the races are (i) deterministic, (ii) probabilistic within the same class, and (iii) probabilistic across classes. In the first group, the player who wins the majority of races receives a one-unit payoff. For this group we show analytically that the three different games with non-singular payoff matrices have the same solution where each player has a unique optimal mixed strategy with equal probabilities. For the second group of games where the payoff to a player is the total number of races his horses have won, we use linear programming with non-numeric data to show that the solution of the three games are mixed strategies given as a convex combination of two extreme points. We invoke results from information theory to prove that to maximize the opponent's "entropy" the players should use the equal probability mixed strategy that was found for the one-unit games.

Keywords: Chinese horse race problem; Probability; Constant-sum game; Mixed strategy, Non-numeric linear programming

1. Introduction

In Chinese history there is a period known as "Spring-Autumn" (770–403 BC) during which China was not a unified empire but consisted of a group of small independent states with conflicting interests. Historical records reveal that during this period more than 200 battles were fought between different

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Chinese states. One of the best-known Chinese philosophers, Sun Tzu (544–496 BC) who wrote *The Art of War* (also known as *The Thirteen Chapters*) was born during this period in the state of Qi. Sun Tzu's work which is recognized as the oldest military treatise and one of the finest of all military classics played an important role in shaping ancient China and helped develop military (and more recently, business) strategies. For further details of Sun Tzu's *Art of War* and his life, see Sun Tzu [1] and Tang [2].

One of the descendents of Sun Tzu, Sun Bin, was also a respected philosopher and a military strategist who witnessed and reported several interesting events in the state of Qi. Sun Bin relates a story known as "Tianji's Horse Race" which is well-known and popular in China. Sun Bin was a friend of General Tianji of the Kingdom of Qi who liked to race horses. One day, the King of Qi wanted to race his horses with those of Tianji's. The King and Tianji each selected three horses with different speed classes. The King's first horse (say, K_1) was faster than all three of Tianji's horses but his second horse (K_2) was only faster than Tianji's second fastest (T_2) and the slowest (T_3) horses. The King's slowest horse (K_3) was only faster than Tianji's slowest horse (T_3). Sun Bin reports that the King and Tianji chose the same class of horses for each race. That is, in the first race the first class horses (K_1 vs. T_1) competed and in the second and third races the second and third class horses (K_2 vs. T_2 and K_3 vs. T_3) competed. Naturally, because Tianji's horse in each class was slower than the King's in the same class, Tianji's horses lost all three races.

Sun Bin offered his friend Tianji some strategic advise to help him win the race. Having learned that the King would continue using the initial winning strategy of racing his horses in the original sequence (K_1,K_2,K_3), Sun Bin suggested Tianji the following strategy: In the first round use the third-class horse (T_3) to compete against the King's first-class horse (K_1); in the second round use the first-class horse (T_1) to compete against the King's second (K_2) and in the third round use the second horse (T_2) to compete against the King's third horse (K_3). The story ends when Tianji uses the strategy suggested by Sun Bin and wins the horse race with one loss and two wins.

In the parlance of modern game theory we would call Sun Bin's advice to Tianji to use the sequence (T_3,T_1,T_2) against the King's fixed strategy of (K_1,K_2,K_3) the "best response" strategy for Tianji. In this case, as the King would race his horses in the same order (K_1,K_2,K_3), Tianji can win 2-to-1 by racing his horses in the order (T_3,T_1,T_2) and receiving the payoff of one unit.

If the King always chooses (K_1,K_2,K_3) and Tianji plays with the optimal best response strategy (T_3,T_1,T_2), Tianji would win every race. Naturally, the King would soon realize that his strategy (K_1,K_2,K_3) is resulting in recurrent losses and would consider alternative strategies to turn the game around. When the King becomes an active player and considers strategies to win the race we encounter a competitive situation which can be analyzed by the tools of game theory. Since each player competes with three horses, in such a game one finds a total of $3! = 6$ strategies (i.e., horse sequences) available to the King and Tianji. Thus, one can formulate a two-person, constant-sum payoff game with a 6×6 payoff matrix whose entries correspond to a pair of payoffs received by the King and Tianji. For example, the payoffs corresponding to the first game with (K_1, K_2, K_3) vs. (T_1, T_2, T_3) would be (1, 0) and the payoffs corresponding to the second game with (K_1, K_2, K_3) vs. (T_3, T_1, T_2) would be (0, 1). Thus, in the simple version of the constant-sum game described above the sum of the King's and Tianji's payoffs is always $c = 1$.

In order to facilitate the analysis of the problem, we can reduce any constant-sum game to a zero-sum game by simply subtracting the constant-sum c from Tianji's payoffs and solve the problem as a zero-sum game in terms of the King's payoffs. This technique of converting a constant-sum game to a zero-sum game by subtracting c from one player's payoffs is a theoretically sound procedure as the players' payoffs are cardinal utilities which are invariant under positive linear transformations; see, e.g., Shubik [3, p. 92] and Straffin [4, pp. 52–53].

Depending on the payoffs received by the players as a result of the three horse races, we analyze two groups of zero-sum games. In the first group the player whose horses win the majority of races receives a payoff of one unit. In this group, we consider three separate cases. In the first case, the outcome of the race is deterministic in the sense that the King's faster horses can beat Tianji's horses in the same or lower classes with certainty; that is, K_i beats T_j with probability 1 for $i, j = 1, 2, 3$ and $i \leqslant j$. In the second case, the outcomes are probabilistic within the same class in the sense that the King's horses beat Tianji's horses in the same class with some probability $p_i \in (0, 1)$, for $i = j$, but the King's horses in faster classes can beat Tianji's horses in the slower classes with certainty. In the third class the outcomes are probabilistic across classes; that is, even the King's horses in faster classes beat Tianji's horses in the same or slower classes with some probability $p_{ij} \in (0, 1)$ for $i, j = 1, 2, 3$ and $i \leqslant j$. For this group of games we compute the optimal mixed strategies analytically when the payoff matrices are non-singular and show that the three different games have the same solution such that each player has a unique optimal mixed strategy with equal probabilities.

We then consider a second group of constant-sum games of horse races where the payoff to a player is the total number of races his horses have won. For example, the payoffs corresponding to the first game described above with (K_1, K_2, K_3) vs. (T_1, T_2, T_3) would be $(3, 0)$ and the payoffs corresponding to the second game with (K_1, K_2, K_3) vs. (T_3, T_1, T_2) would be $(1, 2)$. For this group we again consider the three cases involving deterministic and probabilistic outcomes within and across classes. We use results from the theory of linear programming to show that the solutions of three games in the second group involve infinitely many optimal mixed strategies as a convex combination of two distinct strategies. To analyze the games with general payoff matrices with non-numeric entries, we manipulate the final simplex tableau symbolically to check for optimality. We then answer the question of which one of the infinitely many alternative solutions to employ by using results from information theory. The approach we use to solve a linear programming problem with non-numeric data and our use of information theoretic concepts to choose from among infinitely many alternative optimal solutions appear to be original contributions.

Our models were motivated by the tournament of three races between the King's and Tianji's horses. However, we should note that any tournament where a coach/manager must choose the sequence of players to compete against the opposing team's players can also be modelled using the methodology presented in this paper. For example, tournaments for the game known as "Go" that is especially popular in Asia are played with teams consisting of 5 players. Similar to the first group of horse races with one-unit payoff we analyze in this paper, the team that wins the majority of games wins the tournament. Thus, one can use the game-theoretical approach of this paper to determine the sequence of players to choose in any team in a Go tournament. (For additional details on Go, see `http://gobase.org/games/nn/`.) We should also mention that in addition to game theory, other operations research techniques such as dynamic programming, Markov processes and simulation have also found applications in sports. For a review of these applications see Gerchak [5], Ladany and Machol [6], and Machol et al. [7].

The structure of the paper is organized as follows. Section 2 focuses on our analyses of the first group of one-unit payoff games. For each game, we find that each player has a unique optimal mixed strategy with equal probabilities when the payoff matrix is non-singular. For singular payoff matrices we use linear programming and find two alternative optima. In Section 3 we examine the group of three-unit payoff games. In this section we use linear programming and find the optimal mixed strategies for both players as a convex combination of two alternative strategies. This section also includes a discussion of information-theoretic concept of entropy maximization that leads to a unique strategy among infinitely many strategies for the players. In Section 4, we present our concluding remarks and suggestions for future research topics.

2. Horse races with one-unit payoff

In this section, we formulate and analyze a group of three games with one-unit payoff for the winner. We denote the two players by K (King Qi) and T (Tianji). Three horses owned, respectively, by K and T are represented by (K_1, K_2, K_3) and (T_1, T_2, T_3) where the subscripts refer to the horse class, that is, K_i (or, T_i) $i=1,2,3$ is defined as player K's (or T's) horse in the ith class. Furthermore, we assume that the horse in the first class is the fastest and the horse in the third class is the slowest. Denoting the relation "faster than" by the symbol "$\succ$", we have, for each player, $K_1 \succ K_2 \succ K_3$ and $T_1 \succ T_2 \succ T_3$.

As described in Section 1, player K (or T) makes a decision on the sequence of his horses for the three races. Thus, for each player there are $3!=6$ horse sequences (strategies): $S_1^P=(1,2,3)$, $S_2^P=(1,3,2)$, $S_3^P=(2,1,3)$, $S_4^P=(2,3,1)$, $S_5^P=(3,1,2)$, and $S_6^P=(3,2,1)$ for the player $P=\{K,T\}$. For example, the sequence $S_5^P=(3,1,2)$ represents using the slowest horse in the first race, fastest horse in the second race and the second fastest horse in the last race (as did Tianji after following Sun Bin's advice). Thus, the strategy sets for players K and T, denoted, respectively, by $\mathbf{S}^K$ and $\mathbf{S}^T$, consist of the six strategies and are given by the *ordered* lists $\mathbf{S}^K=(S_1^K,\ldots,S_6^K)$ and $\mathbf{S}^T=(S_1^T,\ldots,S_6^T)$. For all games we define p_{ij} as the probability that the King wins when his horse K_i competes with Tianji's horse T_j, for $i,j=1,2,3$.

2.1. One-unit payoff game with deterministic outcomes

We now restrict our attention to the one-unit payoff game with deterministic outcomes where K_i beats T_j, $i,j=1,2,3$ for $i \leqslant j$ with certainty. In this case, the win probability p_{ij} $(i,j=1,\ldots,6)$ is either zero or one, i.e., the probability matrix is given by

$$\mathbf{P}_1=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

As the outcome of a race is deterministic for this case, we compute player K's and T's payoffs corresponding to each pair of strategies by simply counting the number of times each player's horses win and awarding a unit payoff to the player with a majority of wins. For example, suppose that K's strategy is $S_1^K=(1,2,3)$ and T's strategy is $S_2^T=(1,3,2)$. Since player K certainly wins in the first and second rounds and loses in the third round, he beats T with two wins and one loss and thus receives a payoff of one unit while T gets nothing. Computing the payoffs for the other pairs of strategies, we obtain the payoff matrix (in terms of K's payoffs) for this game as

$$\mathbf{A}_1=\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \tag{2}$$

where the rows correspond to the King's strategies $(S_1^K, \ldots, S_6^K)$ and the columns to Tianji's strategies $(S_1^T, \ldots, S_6^T)$. This matrix is non-singular as its determinant is $\det(\mathbf{A}_1) = -5 \neq 0$.

We now present a Lemma which is used to compute the optimal mixed strategies for each player.

Lemma 1 (*Dresher [8, p. 43]*). *Suppose all pure strategies for each player in a two-person zero-sum matrix game are active (i.e., no dominance or saddle point exists in the game) and the matrix of the game is square and non-singular. Then a unique optimal mixed strategy for each player can be computed using*

$$\mathbf{x}^* = \frac{(\mathbf{A}')^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{A}^{-1}\mathbf{1}} \quad \text{and} \quad \mathbf{y}^* = \frac{\mathbf{A}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{A}^{-1}\mathbf{1}}, \tag{3}$$

where $\mathbf{A}$ *is an* $n \times n$ *non-singular matrix of the game;* $\mathbf{1} = (1, 1, \ldots, 1)'$ *is an* $n \times 1$ *column vector of* 1*'s, and the column vectors* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are the optimal mixed strategies for the players with row and column strategies in the matrix game. Additionally, value of the game to the row player using* $\mathbf{x}^*$ *is given by* $u = (\mathbf{1}'\mathbf{A}^{-1}\mathbf{1})^{-1}$.

In our problem with the square and non-singular payoff matrix $\mathbf{A}_1$ given by (2) there are no dominated strategies nor any saddle points implying that the optimal mixed strategies can be calculated using (3).

Proposition 1. *In the one-unit payoff game with deterministic outcomes and payoff matrix* $\mathbf{A}_1$, *the two players have the same unique optimal mixed strategies with equal probabilities of* $\frac{1}{6}$. *In addition, value of the game to the King is*

$$u_1 = \frac{5}{6}$$

and to Tianji is $v_1 = \frac{1}{6}$.

Proof. Since no saddle point or dominance exists in this game, all pure strategies of each player are active. To compute the mixed strategy probabilities we use the formulas in Lemma 1 with the square 6×6 matrix $\mathbf{A}_1$ given by (2) and $\mathbf{1} = (1, 1, 1, 1, 1, 1)'$. Inverting $\mathbf{A}_1$ gives

$$\mathbf{A}_1^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -4 \\ 1 & 1 & 1 & 1 & -4 & 1 \\ -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Substituting $\mathbf{A}_1$, $\mathbf{A}_1^{-1}$ and $\mathbf{1}$ into (3), we find K's and T's unique optimal mixed strategies $\mathbf{x}^*$ and $\mathbf{y}^*$, respectively, as follows:

$$\mathbf{x}^* = \frac{(\mathbf{A}_1')^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{A}_1^{-1}\mathbf{1}} = \left(\tfrac{6}{5}\right)^{-1}\left(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\right)' = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)',$$

$$\mathbf{y}^* = \frac{\mathbf{A}_1^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{A}_1^{-1}\mathbf{1}} = \left(\tfrac{6}{5}\right)^{-1}\left(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\right)' = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)',$$

which imply that both players choose each of six pure strategies with equal probability. Additionally, value of the game for the King is obtained as $u_1 = (\mathbf{1}'\mathbf{A}_1^{-1}\mathbf{1})^{-1} = \frac{5}{6}$ with the value of the game for Tianji being $v_1 = c - u_1 = 1 - \frac{5}{6} = \frac{1}{6}$. $\square$

Remark 1. The result in Proposition 1 shows that both the King and Tianji should play their mixed strategies completely randomly with equal probabilities which could be done by rolling a fair die. At this point it may be interesting to examine this result in the context of information theory. Consider, for example, Tianji and the uncertainty he faces when he wants to "guess" what the King will do. Let $\mathbf{x} = (x_1, \dots, x_6)'$ be the mixed strategy that will be played by the King with $\sum_{k=1}^{6} x_k = 1$ and define $H^T(\mathbf{x}) = -\sum_{k=1}^{6} x_k \log(x_k)$ as a measure of uncertainty (i.e., "entropy") faced by Tianji when the King chooses his mixed strategy $(x_1, \dots, x_6)$. This definition of measure of uncertainty for discrete random variables was first given by Shannon [9]; see also Ash [10, p. 24].

Maximizing $H^T(\mathbf{x})$ subject to $\sum_{k=1}^{6} x_k = 1$ we find $\mathbf{x}^* = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)'$ which was the optimal mixed strategy for the King. (For this $\mathbf{x}^*$, we find the maximum uncertainty as $H^T(\mathbf{x}^*) \approx 1.79$.) Thus, in this game Tianji faces the maximum amount of uncertainty (entropy) if he attempts to guess the King's strategy. Naturally, since Tianji's mixed strategy is the same as that of the King's, the latter also faces the maximum uncertainty when he tries to guess Tianji's strategy.

2.2. *One-unit payoff game with probabilistic outcomes for the same classes*

We now consider the second case where the outcomes are probabilistic within the same class and again find the optimal mixed strategies and the value of the game using Lemma 1. We now assume that

$$\mathbf{P}_2 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} p_1 & 1 & 1 \\ 0 & p_2 & 1 \\ 0 & 0 & p_3 \end{bmatrix}, \tag{4}$$

where $p_i \in (0, 1)$, $i = 1, 2, 3$ denotes the probability that horse K_i beats horse T_i.

This definition provides more generality by allowing the win probability p_i to take any value in the interval $(0, 1)$. However, for this problem the payoffs are computed as expectations since the result of a race is not known in advance.

To calculate the King's expected payoff E_{kt}^K for a given pair of strategies (S_k^K, S_t^T), with $S_k^K = (k_1, k_2, k_3)$ and $S_t^T = (t_1, t_2, t_3)$ for strategies $k, t = 1, \dots, 6$, we use the formula

$$\begin{aligned} E_{kt}^K &= 1 \cdot \Pr(K \text{ wins}) + 0 \cdot \Pr(K \text{ loses}) \\ &= 1 \cdot [\Pr(K \text{ wins all three rounds}) + \Pr(K \text{ wins any two rounds})] \\ &= p_{k_1 t_1} p_{k_2 t_2} p_{k_3 t_3} + p_{k_1 t_1} p_{k_2 t_2}(1 - p_{k_3 t_3}) + p_{k_1 t_1} p_{k_3 t_3}(1 - p_{k_2 t_2}) + p_{k_2 t_2} p_{k_3 t_3}(1 - p_{k_1 t_1}). \end{aligned} \tag{5}$$

To illustrate the expectation formula (5), we compute the expected payoff to K for two different pairs of strategies. Consider first the case where K and T choose the pure strategies $S_1^K = (1, 2, 3)$ and $S_2^T = (1, 3, 2)$, respectively. For this pair the expected payoff to K, using (4), is

$$\begin{aligned} E_{12}^K &= p_{11} p_{23} p_{32} + [p_{11} p_{23}(1 - p_{32}) + p_{11} p_{32}(1 - p_{23}) + p_{23} p_{32}(1 - p_{11})] \\ &= p_1 \cdot 1 \cdot 0 + [p_1 \cdot 1 \cdot 1 + p_1 \cdot 0 \cdot 0 + 1 \cdot 0 \cdot (1 - p_1)] \\ &= p_1. \end{aligned}$$

When K and T choose the pure strategies $S_3^K = (2, 1, 3)$ and $S_3^T = (2, 1, 3)$, the expected payoff is found as

$$\begin{aligned} E_{33}^K &= p_{11}p_{22}p_{33} + [p_{11}p_{22}(1-p_{33}) + p_{11}p_{33}(1-p_{22}) + p_{22}p_{33}(1-p_{11})] \\ &= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3. \end{aligned}$$

Applying formula (5) to compute E_{kt}^K for each pair of pure strategies S_k^K and S_t^T $(k, t = 1, \ldots, 6)$ results in the following *symbolic* matrix $\mathbf{A}_2$ in terms of K's expected payoffs

$$\mathbf{A}_2 = \begin{bmatrix} d(\mathbf{p}) & p_1 & p_3 & 1 & 0 & p_2 \\ p_1 & d(\mathbf{p}) & 1 & p_3 & p_2 & 0 \\ p_3 & 0 & d(\mathbf{p}) & p_2 & p_1 & 1 \\ 0 & p_3 & p_2 & d(\mathbf{p}) & 1 & p_1 \\ 1 & p_2 & p_1 & 0 & d(\mathbf{p}) & p_3 \\ p_2 & 1 & 0 & p_1 & p_3 & d(\mathbf{p}) \end{bmatrix}, \tag{6}$$

where $\mathbf{p} = (p_1, p_2, p_3)'$ and the diagonal elements are $d(\mathbf{p}) = p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3$. The inverse $\mathbf{A}_2^{-1}$ of the symbolic matrix $\mathbf{A}_2$ can be computed symbolically and it is given in Appendix A.

Lemma 2. *The diagonal element* $d(\mathbf{p})$ *takes values in the interval* $(0, 1)$ *for any* $p_i \in (0, 1)$, $i = 1, 2, 3$.

Proof. First note that

$$\begin{aligned} d(\mathbf{p}) &= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3 \\ &= p_1p_2(1-p_3) + p_1p_3(1-p_2) + p_2p_3 > 0 \end{aligned}$$

since the probabilities take values in the interval $(0, 1)$. By inspection, or by using a nonlinear programming software such as LINGO [11], we maximize $d(\mathbf{p})$ subject to $0^+ \leqslant p_i \leqslant 1^-$, $i = 1, 2, 3$ and find $d_{\max}(\mathbf{p}^*) = 1^-$. Similarly, minimizing $d(\mathbf{p})$ subject to the same bounds we find that $d_{\min}(\mathbf{p}^*) = 0^+$. Thus, $d(\mathbf{p}) \in (0, 1)$. □

Lemma 3. *The determinant of the* $\mathbf{A}_2$ *matrix is zero when*

$$1 + d(\mathbf{p}) = \sum_{i=1}^{3} p_i. \tag{7}$$

Proof. Computing the determinant we find

$$\det(\mathbf{A}_2) = \left[1 + d(\mathbf{p})^2 + 2p_1p_2p_3 - \sum_{i=1}^{3} p_i^2\right]^2 \left\{[1 + d(\mathbf{p})]^2 - \left(\sum_{i=1}^{3} p_i\right)^2\right\}. \tag{8}$$

Here, the first term assumes values in the open interval $(0, 1)$, thus it is positive. The second term assumes values in the interval $(-5, 1)$ and hence it may be zero for some combination of (p_1, p_2, p_3) values when $[1 + d(\mathbf{p})]^2 = \left(\sum_{i=1}^{3} p_i\right)^2$. This proves the Lemma. □

Remark 2. The singularity condition (7) for $\mathbf{A}_2$ has an interesting probabilistic interpretation: First define $\mathscr{E}_i$ as the event that K_i beats T_i for $i = 1, 2, 3$. Now, rearranging condition (7) we obtain $1 - p_1p_2p_3 = p_1 + p_2 + p_3 - p_1p_2 - p_1p_3 - p_2p_3 + p_1p_2p_3$, which is

$$1 - \Pr(\mathscr{E}_1 \cap \mathscr{E}_2 \cap \mathscr{E}_3) = \Pr(\mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3),$$
$$\Pr(K \text{ loses at least one round}) = \Pr(K \text{ wins at least one round}).$$

The last condition can also be written as $1 - \Pr(K \text{ wins all rounds}) = 1 - \Pr(K \text{ loses all rounds})$, or,

$$\Pr(K \text{ wins all rounds}) = \Pr(K \text{ loses all rounds}).$$

This condition is satisfied when, e.g., $p_1 = p_2 = p_3 = \frac{1}{2}$.

With the payoff matrix $\mathbf{A}_2$ in (6) and the column vector $\mathbf{1} = (1, 1, 1, 1, 1, 1)'$, we find the optimal mixed strategies for both players, as shown in the next Proposition.

Proposition 2. *Consider the one-unit payoff game of probabilistic outcomes for the same class. When the payoff matrix* (6) *is non-singular, the two players have the same unique optimal mixed strategies with equal probabilities, i.e.,* $\mathbf{x}^* = \mathbf{y}^* = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})'$. *Moreover, the value of the game to the King is*

$$u_2 = \frac{1}{6}\left[1 + d(\mathbf{p}) + \sum_{i=1}^{3} p_i\right] \tag{9}$$

with $\frac{1}{6} < u_2 < \frac{5}{6}$ *and the value of the game to Tianji is* $v_2 = \frac{5}{6} - \frac{1}{6}[d(\mathbf{p}) + \sum_{i=1}^{3} p_i]$.

Proof. We note that this game has neither a saddle point nor any of the pure strategies are dominated. Assuming that $\mathbf{A}_2$ is non-singular, i.e., that $1 + d(\mathbf{p}) \neq \sum_{i=1}^{3} p_i$, we compute the symbolic inverses of $\mathbf{A}_2$ and $\mathbf{A}_2'$, using the computer algebra system Maple [12,13]. (See Appendix A for the inverse of $\mathbf{A}_2$). Again using Lemma 1, we find the mixed strategy probabilities to obtain $\mathbf{x}^* = \mathbf{y}^* = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})'$.

The value of the game to the King is computed from $u_2 = (\mathbf{1}'\mathbf{A}_2^{-1}\mathbf{1})^{-1}$ which gives $u_2 = \frac{1}{6}[1 + d(\mathbf{p}) + \sum_{i=1}^{3} p_i]$. We now examine the upper and lower bounds on the value of the game to King: Maximizing u_2 subject to $0^+ \leqslant p_i \leqslant 1^-$, $i = 1, 2, 3$, we find the optimal solution as $\mathbf{p}^* = (1^-, 1^-, 1^-)'$ with $\max\ u_2 = (\frac{5}{6})^-$. This was the value of the game when outcomes were deterministic as was shown in Proposition 1. Next, minimizing u_2 subject to the same constraints gives the optimal solution as $\mathbf{p}^* = (0^+, 0^+, 0^+)'$ and $\min\ u_2 = (\frac{1}{6})^+$. Note that when $\mathbf{p}^* = (1^-, 1^-, 1^-)'$, the game reduces to the one with deterministic outcomes as discussed in Section 2.1. □

When the $\mathbf{A}_2$ matrix is singular, the method proposed in Lemma 1 cannot be used in which case we solve the problem using linear programming. It would be unlikely to have a singular $\mathbf{A}_2$ matrix (which would require $1 + d(\mathbf{p}) = \sum_{i=1}^{3} p_i$ as indicated in Lemma 3). However, for the sake of completeness—and for use in subsequent sections—we now present a procedure for solving the game when $\mathbf{A}_2$ is singular.

It is well-known that any two-person zero-sum game can be solved using linear programming (LP); see, e.g., Dantzig [14, Chapter 13], Wang [15] and Zionts [16, Chapter 10]. Consider now the problem faced by the King who must determine his mixed strategy probabilities $\mathbf{x}$. The idea is to assure that *for any* strategy chosen by Tianji the King maximizing u_2 while receiving at least u_2. As an example, we choose $p_1 = p_2 = p_3 = \frac{1}{2}$ so that $d(\mathbf{p}) = \frac{1}{2}$ in which case the $\mathbf{A}_2$ matrix becomes singular and the LP formulation of the game is obtained as

$$\begin{array}{ll} \max & z = u_2 \\ \text{s.t.} & \mathbf{A}_2'\mathbf{x} \geqslant \mathbf{1}u_2 \\ & \mathbf{1}'\mathbf{x} = 1 \end{array} \tag{10}$$

with $\mathbf{x} \geqslant \mathbf{0}$ and $u_2 \geqslant 0$.

Solving this LP (and using the procedure that will be presented later in Section 3.2 and Appendix A) we find infinitely many alternative solutions given by $\mathbf{x}^* = \lambda\mathbf{x}_a + (1-\lambda)\mathbf{x}_b$, $\lambda \in [0, 1]$, and $\mathbf{y}^* = \mu\mathbf{y}_a + (1-\mu)\mathbf{y}_b$, $\mu \in [0, 1]$ where $\mathbf{x}_a = \mathbf{y}_a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right)'$ and $\mathbf{x}_b = \mathbf{y}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$. Note that when $\lambda = \mu = \frac{1}{2}$, we obtain $\mathbf{x}^* = \mathbf{y}^* = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)'$—the same result found in Proposition 2 when $\mathbf{A}_2$ was non-singular.

2.3. One-unit payoff game with probabilistic outcomes across classes

We now consider the most general problem with a unit payoff by assuming that the outcomes of the game are probabilistic across classes. In other words, we assume that the probability that the King's *i*th class horse K_i will beat Tianji's *j*th class horse T_j is p_{ij} for $i, j = 1, 2, 3$. Thus, the probability matrix is now defined as

$$\mathbf{P}_3 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}. \tag{11}$$

Here, although our analysis is valid for any value of $p_{ij} \in (0, 1)$, we make the plausible assumption that

$$\begin{array}{ll} p_{ij} < p_{i,j+1} & \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, \\ p_{ij} > p_{i+1,j} & \text{for } j = 1, 2, 3 \text{ and } i = 1, 2. \end{array}$$

That is, the probabilities in each row are monotone increasing in *j* and the probabilities in each column are monotone decreasing in *i*. For example, the King's fastest horse K_1 has a better chance of beating Tianji's second fastest horse T_2 than Tianji's fastest horse T_1, and an even better chance of beating Tianji's slowest horse T_3.

With the probabilities $p_{ij} \in (0, 1)$ we obtain the most general King–Tianji horse race game with three horses for which the expectations E_{kt}^K for $k, t = 1, \ldots, 6$ can still be calculated using formula (5). For example, E_{12}^K is found as

$$\begin{aligned} E_{12}^K &= p_{11}p_{23}p_{32} + [p_{11}p_{23}(1 - p_{32}) + p_{11}p_{32}(1 - p_{23}) + p_{23}p_{32}(1 - p_{11})] \\ &= p_{11}p_{23} + p_{23}p_{32} + p_{11}p_{32} - 2p_{11}p_{23}p_{32}. \end{aligned}$$

Computing the remaining E_{kt}^K, for $k, t = 1, 2, \ldots, 6$ we find the payoff matrix $\mathbf{A}_3$ for this game as

$$\mathbf{A}_3 = \begin{bmatrix} e_1(\hat{\mathbf{p}}) & e_2(\hat{\mathbf{p}}) & e_3(\hat{\mathbf{p}}) & e_4(\hat{\mathbf{p}}) & e_5(\hat{\mathbf{p}}) & e_6(\hat{\mathbf{p}}) \\ e_2(\hat{\mathbf{p}}) & e_1(\hat{\mathbf{p}}) & e_4(\hat{\mathbf{p}}) & e_3(\hat{\mathbf{p}}) & e_6(\hat{\mathbf{p}}) & e_5(\hat{\mathbf{p}}) \\ e_3(\hat{\mathbf{p}}) & e_5(\hat{\mathbf{p}}) & e_1(\hat{\mathbf{p}}) & e_6(\hat{\mathbf{p}}) & e_2(\hat{\mathbf{p}}) & e_4(\hat{\mathbf{p}}) \\ e_5(\hat{\mathbf{p}}) & e_3(\hat{\mathbf{p}}) & e_6(\hat{\mathbf{p}}) & e_1(\hat{\mathbf{p}}) & e_4(\hat{\mathbf{p}}) & e_2(\hat{\mathbf{p}}) \\ e_4(\hat{\mathbf{p}}) & e_6(\hat{\mathbf{p}}) & e_2(\hat{\mathbf{p}}) & e_5(\hat{\mathbf{p}}) & e_1(\hat{\mathbf{p}}) & e_3(\hat{\mathbf{p}}) \\ e_6(\hat{\mathbf{p}}) & e_4(\hat{\mathbf{p}}) & e_5(\hat{\mathbf{p}}) & e_2(\hat{\mathbf{p}}) & e_3(\hat{\mathbf{p}}) & e_1(\hat{\mathbf{p}}) \end{bmatrix}, \tag{12}$$

where $\hat{\mathbf{p}} = (p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}, p_{31}, p_{32}, p_{33})'$ and

$$e_1(\hat{\mathbf{p}}) = p_{11}p_{22} + p_{22}p_{33} + p_{11}p_{33} - 2p_{11}p_{22}p_{33} > 0, \tag{13}$$

$$e_2(\hat{\mathbf{p}}) = p_{11}p_{23} + p_{23}p_{32} + p_{11}p_{32} - 2p_{11}p_{23}p_{32} > 0, \tag{14}$$

$$e_3(\hat{\mathbf{p}}) = p_{12}p_{21} + p_{21}p_{33} + p_{12}p_{33} - 2p_{12}p_{21}p_{33} > 0, \tag{15}$$

$$e_4(\hat{\mathbf{p}}) = p_{12}p_{23} + p_{23}p_{31} + p_{12}p_{31} - 2p_{12}p_{23}p_{31} > 0, \tag{16}$$

$$e_5(\hat{\mathbf{p}}) = p_{13}p_{21} + p_{21}p_{32} + p_{13}p_{32} - 2p_{13}p_{21}p_{32} > 0, \tag{17}$$

$$e_6(\hat{\mathbf{p}}) = p_{13}p_{22} + p_{22}p_{31} + p_{13}p_{31} - 2p_{13}p_{22}p_{31} > 0. \tag{18}$$

Fortunately, even in this general case it is still possible to use the results in Lemma 1 to compute the mixed strategies and the value of the game when $\mathbf{A}_3$ is non-singular. Before we present the solution for mixed strategies we give conditions under which $\mathbf{A}_3$ is singular.

Lemma 4. *The determinant of the* $\mathbf{A}_3$ *matrix is zero either when*

$$e_1(\hat{\mathbf{p}}) + e_4(\hat{\mathbf{p}}) + e_5(\hat{\mathbf{p}}) = e_2(\hat{\mathbf{p}}) + e_3(\hat{\mathbf{p}}) + e_6(\hat{\mathbf{p}}), \tag{19}$$

or, *when*

$$\begin{aligned}&[e_1(\hat{\mathbf{p}}) - e_4(\hat{\mathbf{p}})]^2 + [e_1(\hat{\mathbf{p}}) - e_5(\hat{\mathbf{p}})]^2 + [e_4(\hat{\mathbf{p}}) - e_5(\hat{\mathbf{p}})]^2 \\ &\quad = [e_2(\hat{\mathbf{p}}) - e_3(\hat{\mathbf{p}})]^2 + [e_2(\hat{\mathbf{p}}) - e_6(\hat{\mathbf{p}})]^2 + [e_3(\hat{\mathbf{p}}) - e_6(\hat{\mathbf{p}})]^2.\end{aligned} \tag{20}$$

Proof. The determinant of the $\mathbf{A}_3$ matrix is computed as

$$\begin{aligned}\det(\mathbf{A}_3) = {}& \left[\sum_{m=1}^{6} e_m(\hat{\mathbf{p}})\right] \times \tilde{e}(\hat{\mathbf{p}}) \times [\tilde{E}(\hat{\mathbf{p}}) - e_1(\hat{\mathbf{p}})e_4(\hat{\mathbf{p}}) - e_1(\hat{\mathbf{p}})e_5(\hat{\mathbf{p}}) - e_4(\hat{\mathbf{p}})e_5(\hat{\mathbf{p}}) \\ & + e_2(\hat{\mathbf{p}})e_3(\hat{\mathbf{p}}) + e_2(\hat{\mathbf{p}})e_6(\hat{\mathbf{p}}) + e_3(\hat{\mathbf{p}})e_6(\hat{\mathbf{p}})]^2,\end{aligned} \tag{21}$$

where

$$\begin{aligned}\tilde{e}(\hat{\mathbf{p}}) &\equiv e_1(\hat{\mathbf{p}}) - e_2(\hat{\mathbf{p}}) - e_3(\hat{\mathbf{p}}) + e_4(\hat{\mathbf{p}}) + e_5(\hat{\mathbf{p}}) - e_6(\hat{\mathbf{p}}), \\ \tilde{E}(\hat{\mathbf{p}}) &\equiv e_1(\hat{\mathbf{p}})^2 - e_2(\hat{\mathbf{p}})^2 - e_3(\hat{\mathbf{p}})^2 + e_4(\hat{\mathbf{p}})^2 + e_5(\hat{\mathbf{p}})^2 - e_6(\hat{\mathbf{p}})^2.\end{aligned}$$

Completing the square inside the third (squared) term in (21), the determinant reduces to

$$\begin{aligned}\det(\mathbf{A}_3) = {}& \left[\sum_{m=1}^{6} e_m(\hat{\mathbf{p}})\right] \times \tilde{e}(\hat{\mathbf{p}}) \times \tfrac{1}{4}\big(\{[e_1(\hat{\mathbf{p}}) - e_4(\hat{\mathbf{p}})]^2 + [e_1(\hat{\mathbf{p}}) - e_5(\hat{\mathbf{p}})]^2 + [e_4(\hat{\mathbf{p}}) - e_5(\hat{\mathbf{p}})]^2\} \\ & - \{[e_2(\hat{\mathbf{p}}) - e_3(\hat{\mathbf{p}})]^2 + [e_2(\hat{\mathbf{p}}) - e_6(\hat{\mathbf{p}})]^2 + [e_3(\hat{\mathbf{p}}) - e_6(\hat{\mathbf{p}})]^2\}\big)^2,\end{aligned}$$

where the first term inside the brackets is positive. Examining the second and the third terms, we obtain the conditions in (19) and (20). □

Proposition 3. *Consider the one-unit payoff game of probabilistic outcomes across classes. When the payoff matrix* (12) *is non-singular, the two players have the same optimal mixed strategies with equal probabilities, i.e.,* $\mathbf{x}^* = \mathbf{y}^* = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)'$. *In addition, the value of the game to the King is*

$$u_3 = \frac{1}{6}\sum_{m=1}^{6} e_m(\hat{\mathbf{p}})$$

with $0 < u_3 < 1$ *and the value of the game to Tianji is* $v_3 = 1 - \frac{1}{6}\sum_{m=1}^{6} e_m(\hat{\mathbf{p}})$ *where* $e_m(\hat{\mathbf{p}})$, $m = 1, \ldots, 6$ *are given in* (13)–(18).

Proof. The proof follows similar lines of arguments as in Proposition 2, thus it will not be repeated here. However, see Appendix B for the inverse of $\mathbf{A}_3$ which is required in the proof of the Proposition. □

Before moving on to the examination of races with three-unit payoffs, we note that when the $\mathbf{A}_3$ matrix is singular, we can use the same LP formulation given by (10). The solution of the LP with singular $\mathbf{A}_2$ replaced by singular $\mathbf{A}_3$ in (10) still gives the same result found in Section 2.2, i.e., where $\mathbf{x}_a = \mathbf{y}_a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right)'$ and $\mathbf{x}_b = \mathbf{y}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$.

As a final remark, consider the "degenerate" case where $p_{ij} = 1$ for all $i, j = 1, 2, 3$, i.e., that King's horses win all three races regardless of which strategy chosen by the players. (This case corresponds to the situation where even the slowest horse owned by the King is faster than the fastest horse owned by Tianji.) Since the $\mathbf{A}_3$ matrix is singular—it consists of all 1's—we solve the problem using LP and find that there are 6 distinct alternative optimal solutions for the King given by the unit vectors $\mathbf{x}_1 = (1, 0, 0, 0, 0, 0)$, $\mathbf{x}_2 = (0, 1, 0, 0, 0, 0), \ldots, \mathbf{x}_6 = (0, 0, 0, 0, 0, 1)$. With this solution the value of the game to King is $u_3 = 1$. Naturally, in this case the convex combination $\mathbf{x} = \sum_{i=1}^{6} \lambda_i \mathbf{x}_i$ is also optimal, and when we set $\lambda_i = \frac{1}{6}$, $i = 1, \ldots, 6$, we find $\mathbf{x}^* = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)'$ which is the solution obtained in all previous problems.

3. Horse races with three-unit payoff

We now assume that the winner in each of the three races receives an award of one-unit. Thus, in this case the maximum possible payoff for the winner in the horse races is three units rather than one unit. For this problem we develop the payoff matrix of the game in terms of *K*'s expected award and analyze the three games with deterministic outcomes, probabilistic outcomes for the same classes and probabilistic outcomes across classes.

3.1. Three-unit payoff game with deterministic outcomes

We now assume, as in Section 2.1, that horse K_i beats horse T_j, $i, j = 1, 2, 3$ for $i \leqslant j$ with certainty, i.e., the probability matrix given by (1) applies. In this setting we calculate *K*'s total payoff corresponding to each pair of strategies adopted by the players as

$$\mathbf{A}_4 = \begin{bmatrix} 3 & 2 & 2 & 2 & 1 & 2 \\ 2 & 3 & 2 & 2 & 2 & 1 \\ 2 & 1 & 3 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 1 & 3 & 2 \\ 2 & 2 & 1 & 2 & 2 & 3 \end{bmatrix}. \tag{22}$$

For example, when K and T, respectively, choose strategies $S_2^K = (1, 3, 2)$ and $S_3^T = (2, 1, 3)$, K wins in the first and third rounds since $K_1 \succ T_2$ and $K_2 \succ T_3$ and loses the second round since $K_3 \prec T_1$. As a result, K wins 2 (as indicated in the second row and third column of $\mathbf{A}_4$) while T receives $3 - 2 = 1$.

For this case we observe that the payoff matrix $\mathbf{A}_4$ has no saddle points, nor any of the strategies are dominated. Moreover, it can be shown that $\det(\mathbf{A}_4) = 0$, i.e., $\mathbf{A}_4$ is singular. Thus Lemma 1 cannot be applied to find the optimal strategies. Hence, we utilize linear programming to analyze this problem as we had done for a special case of the one-unit payoff game in Section 2.2.

As before, we denote K's and T's mixed strategies by $\mathbf{x}=(x_1, \ldots, x_6)'$ and $\mathbf{y}=(y_1, \ldots, y_6)'$, respectively. Using the $\mathbf{A}_4$ matrix, the LP formulation of K's problem is obtained similarly to (10) as

$$\begin{aligned} \max \quad & z = u_4 \\ \text{s.t.} \quad & \mathbf{A}_4'\mathbf{x} \geqslant \mathbf{1}u_4, \\ & \mathbf{1}'\mathbf{x} = 1 \end{aligned} \tag{23}$$

with the usual non-negativity constraints of $\mathbf{x} \geqslant \mathbf{0}$ and $u_4 \geqslant 0$ where $\mathbf{0} = (0, \ldots, 0)'$ is a 6×1 column vector of 0's. After introducing the surplus variables $\mathbf{s} = (s_1, \ldots, s_6)'$ for the first six inequality constraints, we write this problem in the "canonical" form[2] as

$$\begin{aligned} \max \quad & z = \hat{\mathbf{c}}'\hat{\mathbf{x}} \\ \text{s.t.} \quad & \hat{\mathbf{A}}_4\hat{\mathbf{x}} = \hat{\mathbf{b}}, \end{aligned} \tag{24}$$

where

$$\hat{\mathbf{A}}_4 = \begin{bmatrix} \mathbf{A}_4' & -\mathbf{1} & -\mathbf{I} \\ \mathbf{1}' & 0 & \mathbf{0}' \end{bmatrix}_{[7\times 13]}, \quad \hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ u_4 \\ \mathbf{s} \end{bmatrix}_{[13\times 1]}, \quad \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}_{[7\times 1]}, \quad \hat{\mathbf{c}} = \begin{bmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \end{bmatrix}_{[13\times 1]},$$

and $\mathbf{I}_{[7\times 7]}$ is the identity matrix.

The next Proposition presents the complete solution for this game.

Proposition 4. *In the three-unit payoff game with deterministic outcomes, the optimal mixed strategies for K and T are*, $\mathbf{x}^* = \lambda\mathbf{x}_a + (1-\lambda)\mathbf{x}_b$, $\lambda \in [0, 1]$ *and* $\mathbf{y}^* = \mu\mathbf{y}_a + (1-\mu)\mathbf{y}_b$, $\mu \in [0, 1]$, *respectively, where* $\mathbf{x}_a = \mathbf{y}_a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right)'$ *and* $\mathbf{x}_b = \mathbf{y}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$. *Moreover, value of the game for K is*

$$u_4 = 2$$

and for T the value is $v_4 = 1$.

[2] In this problem the proper canonical form would involve introducing seven artificial variables for each of the equality constraints in order to generate the initial basic feasible solution. But since we are not interested in the initial solution and the optimal solution always exists for the LP formulation of the game problems, we ignore the artificial variables in our version of the "canonical" form.

Proof. To prove the Proposition, we first solve problem (24) using a linear programming solver such as LINDO [17] which gives

VARIABLE	x_1	x_2	x_3	x_4	x_5	x_6	u_4	s_1	s_2	s_3	s_4	s_5	s_6
OPTIMAL VALUE	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	2	0	0	0	0	0	0
REDUCED COST	0	0	0	0	0	0	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0

(25)

and

ROW	S_1^T	S_2^T	S_3^T	S_4^T	S_5^T	S_6^T	Probability
SLACK/SURPLUS	0	0	0	0	0	0	0
DUAL PRICES	$-\frac{1}{3}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	2

(26)

with the basic vector $\hat{\mathbf{x}}_B = (x_1, x_4, x_5, u_4, s_2, s_3, s_6)' = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 2, 0, 0, 0\right)'$ and the non-basic vector $\hat{\mathbf{x}}_N = (x_2, x_3, x_6, s_1, s_4, s_5)' = (0, \ldots, 0)'$. We write this optimal extreme point solution in terms of the decision variables as $\mathbf{x}_a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right)'$.

For the optimal solution that we have found the basis matrix $\mathbf{B}$ of the basic vector $\hat{\mathbf{x}}_B$ and the matrix $\mathbf{N}$ of non-basic vector $\hat{\mathbf{x}}_N$ are

$$\mathbf{B} = \begin{bmatrix} 3 & 1 & 2 & -1 & 0 & 0 & 0 \\ 2 & 2 & 2 & -1 & -1 & 0 & 0 \\ 2 & 2 & 2 & -1 & 0 & -1 & 0 \\ 2 & 3 & 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 2 & 2 & 2 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}_{[7\times 7]}, \quad \mathbf{N} = \begin{bmatrix} 2 & 2 & 2 & -1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & -1 & 0 \\ 2 & 2 & 2 & 0 & 0 & -1 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}_{[7\times 6]},$$

respectively. Let us denote $\hat{\mathbf{c}}_B$ and $\hat{\mathbf{c}}_N$ as the vector of the objective function coefficients for $\hat{\mathbf{x}}_B$ and $\hat{\mathbf{x}}_N$, respectively. Then, in a maximization problem the components of the "evaluator" vector

$$\hat{\mathbf{c}}_N' - \hat{\mathbf{c}}_B'\mathbf{B}^{-1}\mathbf{N} = (0, 0, 0, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3})$$

corresponding to the non-basic variables $(x_2, x_3, x_6, s_1, s_4, s_5)$ indicate the amount by which the objective would improve if a particular non-basic variable were to become basic; see, Zionts [16, Chapter 3]. (The negatives of these values are known as the "reduced costs.") Thus, the above results indicate the presence of multiple and degenerate optimal solutions since some non-basic variables (i.e., x_2, x_3, and x_6) can be made basic without affecting the objective function value as the reduced cost for these variables is zero. Also note that the non-basic variables (s_1, s_4, s_5) would never be candidates for entering into the basis as their $\hat{\mathbf{c}}_N' - \hat{\mathbf{c}}_B'\mathbf{B}^{-1}\mathbf{N}$ components are all negative (and, their reduced costs are all positive).

In general, the constraints can be written in terms of the basic and non-basic vector $(\hat{\mathbf{x}}_B, \hat{\mathbf{x}}_N)$ as $\mathbf{B}\hat{\mathbf{x}}_B + \mathbf{N}\hat{\mathbf{x}}_N = \hat{\mathbf{b}}$. Left-multiplying this equation by $\mathbf{B}^{-1}$, the "canonical" form corresponding to the basis matrix $\mathbf{B}$ is obtained as $\hat{\mathbf{x}}_B + \mathbf{B}^{-1}\mathbf{N}\hat{\mathbf{x}}_N = \mathbf{B}^{-1}\hat{\mathbf{b}}$ which reduces to

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1}\hat{\mathbf{b}} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 2, 0, 0, 0\right)'$$

when $\hat{\mathbf{x}}_N = \mathbf{0}$. To determine the other basic extreme point optimal solution we now allow the (x_2, x_3, x_6) variables in the $\hat{\mathbf{x}}_N$ vector to assume nonzero values and solve for $\hat{\mathbf{x}}_B$ to obtain $\hat{\mathbf{x}}_B = \mathbf{B}^{-1}\hat{\mathbf{b}} - \mathbf{B}^{-1}\mathbf{N}\hat{\mathbf{x}}_N$. This gives the basic variables as a function of the non-basic decision variables as follows:

$$\begin{aligned}
x_1 &= \tfrac{1}{3} - \tfrac{1}{3}(x_2 + x_3 + x_6),\\
x_4 &= \tfrac{1}{3} - \tfrac{1}{3}(x_2 + x_3 + x_6),\\
x_5 &= \tfrac{1}{3} - \tfrac{1}{3}(x_2 + x_3 + x_6),\\
u_4 &= 2,\\
s_2 &= x_2 - x_3,\\
s_3 &= x_3 - x_6,\\
s_6 &= -x_2 + x_6.
\end{aligned}$$

Clearly, the (x_2, x_3, x_6) variables in the non-basic vector $\hat{\mathbf{x}}_N = (x_2, x_3, x_6, s_1, s_4, s_5)'$ can be varied in any amount provided that the basic variables in $\hat{\mathbf{x}}_B = (x_1, x_4, x_5, u_4, s_2, s_3, s_6)'$ do not assume negative values. Hence, to keep s_2, s_3, and s_6 non-negative, we must have $x_2 \geqslant x_3$, $x_3 \geqslant x_6$ and $x_6 \geqslant x_2$, which implies $x_2 = x_3 = x_6$ and $s_2 = s_3 = s_6 = 0$.

Since $x_2 = x_3 = x_6$,we can write $x_1 = \frac{1}{3} - x_2$ and note that the entering variable x_2 should be increased as much as possible without driving x_1 negative. This implies $x_2 = \frac{1}{3}$ so that $x_1 = 0$. Thus,$x_2 = x_3 = x_6 = \frac{1}{3}$ and $x_4 = x_5 = 0$,i.e.,the new basic vector is $\hat{\mathbf{x}}_B = (x_2, x_3, x_6, u_4, s_1, s_4, s_5)' = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 2, 0, 0, 0\right)'$ so that the other extreme solution in terms of the decision variables is now obtained as $\mathbf{x}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$. Because a convex combination of two extreme point solutions of an LP is also optimal (see [18, p. 52]),the result follows. Similar arguments show that the optimal strategy for Tianji is also obtained as the convex combination of the two extreme solutions $\mathbf{y}_a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right)'$ and $\mathbf{y}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$. Finally, since $u_4 + v_4 = 3$, we obtain Tianji's value as $v_4 = 1$. □

Remark 3. For this problem with multiple optima, one may wonder about the proper choice of λ (and μ) that determines the convex combination of $\mathbf{x}_a$ and $\mathbf{x}_b$ for the King (and $\mathbf{y}_a$ and $\mathbf{y}_b$ for Tianji). To answer this question, consider again the measure of uncertainty $H^T(\mathbf{x}^*)$ that Tianji would face for a given mixed strategy $\mathbf{x}^* = \lambda\mathbf{x}_a + (1-\lambda)\mathbf{x}_b$, $\lambda \in [0, 1]$ chosen by the King. Substituting $\mathbf{x}^* = \frac{1}{3}(\lambda, 1-\lambda, 1-\lambda, \lambda, \lambda, 1-\lambda)$ in $H^T(\mathbf{x}^*) = -\sum_{k=1}^{6} x_k^* \log(x_k^*)$ we find

$$H^T(\mathbf{x}^*, \lambda) = -\lambda \log\left(\tfrac{1}{3}\lambda\right) - (1-\lambda) \log\left(\tfrac{1}{3}(1-\lambda)\right),$$

which is maximized at $\lambda = \frac{1}{2}$ with $H^T\left(\mathbf{x}^*, \frac{1}{2}\right) \approx 1.79$. Thus, in order to maximize his opponent's uncertainty, the King should use the strategy $\mathbf{x}^* = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)'$.

3.2. Three-unit payoff game with probabilistic outcomes for the same classes

With a three unit payoff, when the outcomes are probabilistic within the same class the probability matrix in (4) applies. For this game, to calculate the King's expected payoff E_{kt}^K for a given pair of strategies $\left(S_k^K, S_t^T\right)$ for $k, t = 1, \ldots, 6$, we use the expected value

$$\begin{aligned} E_{kt}^K &= 1 \cdot \Pr(K \text{ wins in the 1st round}) + 1 \cdot \Pr(K \text{ wins in the 2nd round}) \\ &\quad + 1 \cdot \Pr(K \text{ wins in the 3rd round}) + 0 \cdot \Pr(K \text{ loses all rounds}). \end{aligned} \tag{27}$$

For example, for $S_2^K = (1, 3, 2)$ and $S_4^T = (2, 3, 1)$ we have $E_{24}^K = 1 \cdot 1 + 1 \cdot p_3 + 1 \cdot 0 = 1 + p_3$. Similarly, for $S_1^K = (1, 2, 3)$ and $S_4^T = (2, 3, 1)$ we find $E_{14}^K = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 2$. After computing the expected payoffs for other strategy pairs we obtain the payoff matrix for this game as

$$\mathbf{A}_5 = \begin{bmatrix} g(\mathbf{p}) & g_1(p_1) & g_3(p_3) & 2 & 1 & g_2(p_2) \\ g_1(p_1) & g(\mathbf{p}) & 2 & g_3(p_3) & g_2(p_2) & 1 \\ g_3(p_3) & 1 & g(\mathbf{p}) & g_2(p_2) & g_1(p_1) & 2 \\ 1 & g_3(p_3) & g_2(p_2) & g(\mathbf{p}) & 2 & g_1(p_1) \\ 2 & g_2(p_2) & g_1(p_1) & 1 & g(\mathbf{p}) & g_3(p_3) \\ g_2(p_2) & 2 & 1 & g_1(p_1) & g_3(p_3) & g(\mathbf{p}) \end{bmatrix},$$

where $g(\mathbf{p}) = \sum_{i=1}^3 p_i$, $g_i(p_i) = 1 + p_i$, $i = 1, 2, 3$ and $\mathbf{p} = (p_1, p_2, p_3)'$. It can be shown that $\det(\mathbf{A}_5) = 0$ for any value of $\mathbf{p}$, i.e., $\mathbf{A}_5$ is singular as was $\mathbf{A}_4$; so Lemma 1 cannot be applied to find the optimal strategies. Thus, as in Section 3.1 we utilize linear programming to analyze this problem.

For this problem, similar to (23), the linear programming formulation is given as

$$\begin{aligned} \max \quad & z = u_5 \\ \text{s.t.} \quad & \mathbf{A}_5'\mathbf{x} \geqslant \mathbf{1}u_5, \\ & \mathbf{1}'\mathbf{x} = 1, \end{aligned}$$

where u_5 is the value of the game to the King. This can again be written in the "canonical" form as $\max\ z = \hat{\mathbf{c}}'\hat{\mathbf{x}}$ s.t. $\hat{\mathbf{A}}_5\hat{\mathbf{x}} = \hat{\mathbf{b}}$ where $\hat{\mathbf{A}}_5$ is defined similarly to $\hat{\mathbf{A}}_4$ in Section 3.1.

Proposition 5. *In the three-unit payoff game with probabilistic outcomes for the same classes, the optimal mixed strategies for K and T are,* $\mathbf{x}^* = \lambda\mathbf{x}_a + (1-\lambda)\mathbf{x}_b$, $\lambda \in [0, 1]$ *and* $\mathbf{y}^* = \mu\mathbf{y}_a + (1-\mu)\mathbf{y}_b$, $\mu \in [0, 1]$, *respectively, where* $\mathbf{x}_a = \mathbf{y}_a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right)'$ *and* $\mathbf{x}_b = \mathbf{y}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$. *Moreover, value of the game for K is*

$$u_5 = 1 + \frac{1}{3}\sum_{i=1}^{3} p_i$$

with $1 < u_5 < 2$ *and the value of game to T is* $v_5 = 2 - \frac{1}{3}\sum_{i=1}^3 p_i$.

Proof. The proof of this Proposition requires the solution of a linear programming problem with non-numeric data and it is presented in Appendix C. □

3.3. Three-unit payoff game with probabilistic outcomes across classes

In our last and most general model with a three unit payoff where outcomes are probabilistic across classes, we assume, as in Section 2.3 that the probability that K_i will beat T_j is p_{ij} for $i, j = 1, 2, 3$. For this game the probability matrix is given by (11).

To calculate the King's expected payoff E^K_{kt} for a given pair of strategies (S^K_k, S^T_t) for $k, t = 1, \ldots, 6$, we again use expression (27). For example, for $S^K_2 = (1, 3, 2)$ and $S^T_4 = (2, 3, 1)$ we have $E^K_{24} = 1 \cdot p_{12} + 1 \cdot p_{33} + 1 \cdot p_{21}$. After computing the expected payoffs for other strategy pairs we obtain the payoff matrix for this game as

$$\mathbf{A}_6 = \begin{bmatrix} h_1(\hat{\mathbf{p}}) & h_2(\hat{\mathbf{p}}) & h_3(\hat{\mathbf{p}}) & h_4(\hat{\mathbf{p}}) & h_5(\hat{\mathbf{p}}) & h_6(\hat{\mathbf{p}}) \\ h_2(\hat{\mathbf{p}}) & h_1(\hat{\mathbf{p}}) & h_4(\hat{\mathbf{p}}) & h_3(\hat{\mathbf{p}}) & h_6(\hat{\mathbf{p}}) & h_5(\hat{\mathbf{p}}) \\ h_3(\hat{\mathbf{p}}) & h_5(\hat{\mathbf{p}}) & h_1(\hat{\mathbf{p}}) & h_6(\hat{\mathbf{p}}) & h_2(\hat{\mathbf{p}}) & h_4(\hat{\mathbf{p}}) \\ h_5(\hat{\mathbf{p}}) & h_3(\hat{\mathbf{p}}) & h_6(\hat{\mathbf{p}}) & h_1(\hat{\mathbf{p}}) & h_4(\hat{\mathbf{p}}) & h_2(\hat{\mathbf{p}}) \\ h_4(\hat{\mathbf{p}}) & h_6(\hat{\mathbf{p}}) & h_2(\hat{\mathbf{p}}) & h_5(\hat{\mathbf{p}}) & h_1(\hat{\mathbf{p}}) & h_3(\hat{\mathbf{p}}) \\ h_6(\hat{\mathbf{p}}) & h_4(\hat{\mathbf{p}}) & h_5(\hat{\mathbf{p}}) & h_2(\hat{\mathbf{p}}) & h_3(\hat{\mathbf{p}}) & h_1(\hat{\mathbf{p}}) \end{bmatrix},$$

where $h_1(\hat{\mathbf{p}}) = p_{11} + p_{22} + p_{33}$, $h_2(\hat{\mathbf{p}}) = p_{11} + p_{23} + p_{32}$, $h_3(\hat{\mathbf{p}}) = p_{12} + p_{21} + p_{33}$, $h_4(\hat{\mathbf{p}}) = p_{12} + p_{23} + p_{31}$, $h_5(\hat{\mathbf{p}}) = p_{13} + p_{21} + p_{32}$, and $h_6(\hat{\mathbf{p}}) = p_{13} + p_{22} + p_{31}$. It can be shown that $\mathbf{A}_6$ is singular, thus Lemma 1 cannot be used. We again resort to linear programming to determine the optimal strategies.

For this problem, similar to (23), the linear programming formulation is given as max $z = u_6$ s.t. $\mathbf{A}'_6\mathbf{x} \geqslant \mathbf{1}u_6$ and $\mathbf{1}'\mathbf{x} = 1$ where u_6 is the value of the game to the King. This can be written in the "canonical" form as max $z = \hat{\mathbf{c}}'\hat{\mathbf{x}}$ s.t. $\hat{\mathbf{A}}_6\hat{\mathbf{x}} = \hat{\mathbf{b}}$ where $\hat{\mathbf{A}}_6$ is defined similarly to $\hat{\mathbf{A}}_4$ in Section 3.1.

Proposition 6. *In the three-unit payoff game with probabilistic outcomes across classes, the optimal mixed strategies for K and T are,* $\mathbf{x}^* = \lambda\mathbf{x}_a + (1-\lambda)\mathbf{x}_b$, $\lambda \in [0, 1]$ *and* $\mathbf{y}^* = \mu\mathbf{y}_a + (1-\mu)\mathbf{y}_b$, $\mu \in [0, 1]$, *respectively, where* $\mathbf{x}_a = \mathbf{y}_a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right)'$ *and* $\mathbf{x}_b = \mathbf{y}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$. *Moreover, value of the game for K is*

$$u_6 = \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}$$

with $0 < u_6 < 3$ *and the value to T is* $v_6 = 3 - \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}$.

Proof. The proof of this Proposition requires the solution of a linear programming problem with non-numeric data and it is presented in Appendix D. □

4. Conclusions

In this paper, we presented a game-theoretical analysis of a legendary Chinese horse race between the King of Qi and General Tianji who raced three horses of differing speeds with the property that the King's horses in a given speed class are better than Tianji's horses in the same class. We analyzed two groups of games with different payoffs to the winner. In the first group, the player who wins the majority of three-round races receives a payoff of one unit and the player who loses the races receives nothing. In the second group, the maximum payoff (the "purse") is three units and each player receives as many units as the number of races won by his horses. For each group, we considered three different games identified according to the outcomes of a specific race: (i) deterministic outcome, (ii) probabilistic outcome for the same speed classes, and (iii) probabilistic outcome across speed classes.

Table 1
One-unit payoff games with different number of horses

N	Number of positive x_i^* in the $N! \times 1$ vector $\mathbf{x}^*$	x_i^*	u
3	$3! = 6$	$1/3!$	$5/6 \approx 0.833$
4	$4! = 24$	$1/4!$	$35/48 \approx 0.729$
5	$5! = 120$	$1/5!$	$31/40 \approx 0.775$

Table 2
Multiple-unit payoff games with different number of horses

N	Number of positive x_i^* in the $N! \times 1$ vector $\mathbf{x}^*$	x_i^*	u
3	3	1/3	2
4	4	1/4	2.5
5	5	1/5	3

We showed that both groups of games can be classified as *constant-sum* with six pure strategies which can be solved using the solution techniques for zero-sum games. For all three games in the first group, we proved that there are no optimal pure strategies and that the optimal mixed strategy calls for choosing a probability of $\frac{1}{6}$ for each pure strategy. We also showed that, for the probabilistic outcome games, there are multiple optimal strategies if the payoff matrix is singular. For all three of the second group of games we proved that the payoff matrices are always singular and there are multiple optimal strategies for both players. We solved the games involving general (non-numeric) payoffs by manipulating the final simplex tableau symbolically to check for optimality. We also answered the question of which one of the infinitely many alternative solutions to employ by using results from information theory.

Naturally, the present models can be easily extended to cases where each player races $N > 3$ horses where ties in even number of games result in a 50–50 split of the total payoff. In such a case, each player would have a total of $N!$ possible pure strategies with a payoff matrix of $N!$ rows and $N!$ columns. For the first group of games we have solved several games with deterministic outcomes and obtained the results in Table 1.

Based on the results in Table 1 we would thus conjecture that the optimal mixed strategy for each player would be of the same form as we found in Section 2, i.e.,

$$\mathbf{x}^* = \mathbf{y}^* = \left(\frac{1}{N!}, \frac{1}{N!}, \ldots, \frac{1}{N!}\right)'.$$

For the second group of games with payoffs equal to the number of horses, we believe that optimal mixed strategies will still be of the same form as the one we found in Section 3, i.e., there will be multiple optimal solutions to the resulting linear programming problem with $2N! + 1$ variables (including the $N!$ surplus variables) and $N! + 1$ constraints. We have solved the problem for different values of N and obtained the results in Table 2.

Based on the results in Table 2, we conjecture that for the case with payoffs equal to the number of horses N, the optimal strategy for each player would involve a solution with N positive probabilities each equal to $\frac{1}{N}$, i.e.,

$$\mathbf{x}^* = \mathbf{y}^* = \left(N \text{ components with value } \frac{1}{N}, \text{ and } N! - N \text{ components with value } 0\right)'.$$

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Appendix A. Inverse of the payoff matrix $\mathbf{A}_2$

The inverse of the symbolic matrix $\mathbf{A}_2$ of the one-unit payoff game with probabilistic outcomes is given as

$$\mathbf{A}_2^{-1} = \begin{bmatrix} \phi_2(\mathbf{p}) & \phi_4(\mathbf{p}) & \phi_3(\mathbf{p}) & \phi_6(\mathbf{p}) & \phi_5(\mathbf{p}) & \phi_7(\mathbf{p}) \\ \phi_4(\mathbf{p}) & \phi_2(\mathbf{p}) & \phi_6(\mathbf{p}) & \phi_3(\mathbf{p}) & \phi_7(\mathbf{p}) & \phi_5(\mathbf{p}) \\ \phi_3(\mathbf{p}) & \phi_5(\mathbf{p}) & \phi_2(\mathbf{p}) & \phi_7(\mathbf{p}) & \phi_4(\mathbf{p}) & \phi_6(\mathbf{p}) \\ \phi_5(\mathbf{p}) & \phi_3(\mathbf{p}) & \phi_7(\mathbf{p}) & \phi_2(\mathbf{p}) & \phi_6(\mathbf{p}) & \phi_4(\mathbf{p}) \\ \phi_6(\mathbf{p}) & \phi_7(\mathbf{p}) & \phi_4(\mathbf{p}) & \phi_5(\mathbf{p}) & \phi_2(\mathbf{p}) & \phi_3(\mathbf{p}) \\ \phi_7(\mathbf{p}) & \phi_6(\mathbf{p}) & \phi_5(\mathbf{p}) & \phi_4(\mathbf{p}) & \phi_3(\mathbf{p}) & \phi_2(\mathbf{p}) \end{bmatrix},$$

where

$$\phi_2(\mathbf{p}) = \frac{1}{\phi_1(\mathbf{p})}\left\{d(\mathbf{p})^3 + d(\mathbf{p})^2 - \tfrac{1}{2}[1+d(\mathbf{p})]\sum_{i=1,2;\,j=2,3}(p_i - p_j)^2 + \sum_{i=1}^{3} p_i^2\right\},$$

$$\phi_3(\mathbf{p}) = \frac{1}{\phi_1(\mathbf{p})}\left\{-[1+d(\mathbf{p})^2]p_3 + d(\mathbf{p})(p_1+p_2-p_3) - (p_1p_2 - p_3^2)\sum_{i=1}^{3} p_i\right\},$$

$$\phi_4(\mathbf{p}) = \frac{1}{\phi_1(\mathbf{p})}\left\{-[1+d(\mathbf{p})^2]p_1 + d(\mathbf{p})(p_2+p_3-p_1) - (p_2p_3 - p_1^2)\sum_{i=1}^{3} p_i\right\},$$

$$\phi_5(\mathbf{p}) = \frac{1}{\phi_1(\mathbf{p})}\left\{[1+p_1p_2+p_1p_3+p_2p_3]d(\mathbf{p}) + 1 - \tfrac{1}{2}\sum_{\substack{i=1,2;\,j=2,3\\ i\neq j}}(p_i+p_j)^2\right\},$$

$$\phi_6(\mathbf{p}) = \frac{1}{\phi_1(\mathbf{p})}\{-[1+d(\mathbf{p})][d(\mathbf{p}) - (p_1p_2+p_1p_3+p_2p_3)]\},$$

$$\phi_7(\mathbf{p}) = \frac{1}{\phi_1(\mathbf{p})}\left\{-[1+d(\mathbf{p})^2]p_2 + d(\mathbf{p})(p_1+p_3-p_2) - (p_1p_3 - p_2^2)\sum_{i=1}^{3} p_i\right\},$$

and $\phi_1(\mathbf{p}) = \det(\mathbf{A}_2)$ given in (8) and $d(\mathbf{p}) = p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3$.

Appendix B. Inverse of the payoff matrix $\mathbf{A}_3$

The inverse of the symbolic matrix $\mathbf{A}_3$ of the one-unit payoff game with probabilistic outcomes across classes is given as

$$\mathbf{A}_3^{-1} = \begin{bmatrix} \theta_2(\mathbf{p}) & \theta_4(\mathbf{p}) & \theta_3(\mathbf{p}) & \theta_6(\mathbf{p}) & \theta_5(\mathbf{p}) & \theta_7(\mathbf{p}) \\ \theta_4(\mathbf{p}) & \theta_2(\mathbf{p}) & \theta_6(\mathbf{p}) & \theta_3(\mathbf{p}) & \theta_7(\mathbf{p}) & \theta_5(\mathbf{p}) \\ \theta_3(\mathbf{p}) & \theta_5(\mathbf{p}) & \theta_2(\mathbf{p}) & \theta_7(\mathbf{p}) & \theta_4(\mathbf{p}) & \theta_6(\mathbf{p}) \\ \theta_5(\mathbf{p}) & \theta_3(\mathbf{p}) & \theta_7(\mathbf{p}) & \theta_2(\mathbf{p}) & \theta_6(\mathbf{p}) & \theta_4(\mathbf{p}) \\ \theta_6(\mathbf{p}) & \theta_7(\mathbf{p}) & \theta_4(\mathbf{p}) & \theta_5(\mathbf{p}) & \theta_2(\mathbf{p}) & \theta_3(\mathbf{p}) \\ \theta_7(\mathbf{p}) & \theta_6(\mathbf{p}) & \theta_5(\mathbf{p}) & \theta_4(\mathbf{p}) & \theta_3(\mathbf{p}) & \theta_2(\mathbf{p}) \end{bmatrix},$$

where

$$\theta_2(\mathbf{p}) = \frac{1}{\theta_1(\mathbf{p})}\{-\tfrac{1}{2}e_1[(e_2+e_3)^2+(e_2+e_6)^2+(e_3+e_6)^2] + (e_2e_3+e_2e_6+e_3e_6)(e_4+e_5)+(e_1+e_4+e_5)(e_1^2-e_4e_5)\},$$

$$\theta_3(\mathbf{p}) = \frac{1}{\theta_1(\mathbf{p})}\{-\tfrac{1}{2}e_3[(e_1+e_4)^2+(e_1+e_5)^2+(e_4+e_5)^2] + (e_1e_4+e_1e_5+e_4e_5)(e_2+e_6)+(e_2+e_3+e_6)(e_3^2-e_2e_6)\},$$

$$\theta_4(\mathbf{p}) = \frac{1}{\theta_1(\mathbf{p})}\{-\tfrac{1}{2}e_2[(e_1+e_4)^2+(e_1+e_5)^2+(e_4+e_5)^2] + (e_1e_4+e_1e_5+e_4e_5)(e_3+e_6)+(e_2+e_3+e_6)(e_2^2-e_3e_6)\},$$

$$\theta_5(\mathbf{p}) = \frac{1}{\theta_1(\mathbf{p})}\{-\tfrac{1}{2}e_4[(e_2+e_3)^2+(e_2+e_6)^2+(e_3+e_6)^2] + (e_2e_3+e_2e_6+e_3e_6)(e_1+e_5)+(e_1+e_4+e_5)(e_4^2-e_1e_5)\},$$

$$\theta_6(\mathbf{p}) = \frac{1}{\theta_1(\mathbf{p})}\{-\tfrac{1}{2}e_5[(e_2+e_3)^2+(e_2+e_6)^2+(e_3+e_6)^2] + (e_2e_3+e_2e_6+e_3e_6)(e_1+e_4)+(e_1+e_4+e_5)(e_5^2-e_1e_4)\},$$

$$\theta_7(\mathbf{p}) = \frac{1}{\theta_1(\mathbf{p})}\{-\tfrac{1}{2}e_6[(e_1+e_4)^2+(e_1+e_5)^2+(e_4+e_5)^2] + (e_1e_4+e_1e_5+e_4e_5)(e_2+e_3)+(e_2+e_3+e_6)(e_6^2-e_2e_3)\},$$

and $\theta_1(\mathbf{p}) = \det(\mathbf{A}_3)$ given in (21) with $e_m \equiv e_m(\mathbf{p})$, for $m = 1, \ldots, 6$.

Appendix C. Proof of Proposition 5

Motivated by the optimal solution found for the three-unit payoff game with deterministic outcomes, we choose the basic vector $\hat{\mathbf{x}}_B = (x_1, x_4, x_5, u_4, s_2, s_3, s_6)'$ corresponding to $\mathbf{x}_a$ and test it for optimality.

The basis matrix $\mathbf{B}$ with non-numeric entries corresponding to $\hat{\mathbf{x}}_B$ and the matrix $\mathbf{N}$ with non-numeric entries corresponding to the non-basic vector $\hat{\mathbf{x}}_N = (x_2, x_3, x_6, s_1, s_4, s_5)'$ are given by

$$\mathbf{B} = \begin{bmatrix} g(\mathbf{p}) & 1 & 2 & -1 & 0 & 0 & 0 \\ g_1(p_1) & g_3(p_3) & g_2(p_2) & -1 & -1 & 0 & 0 \\ g_3(p_3) & g_2(p_2) & g_1(p_1) & -1 & 0 & -1 & 0 \\ 2 & g(\mathbf{p}) & 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & g(\mathbf{p}) & -1 & 0 & 0 & 0 \\ g_2(p_2) & g_1(p_1) & g_3(p_3) & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} g_1(p_1) & g_3(p_3) & g_2(p_2) & -1 & 0 & 0 \\ g(\mathbf{p}) & 1 & 2 & 0 & 0 & 0 \\ 2 & g(\mathbf{p}) & 1 & 0 & 0 & 0 \\ g_3(p_3) & g_2(p_2) & g_1(p_1) & 0 & -1 & 0 \\ g_2(p_2) & g_1(p_1) & g_3(p_3) & 0 & 0 & -1 \\ 1 & 2 & g(\mathbf{p}) & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

respectively. Using these matrices the basic vector is computed as

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1}\hat{\mathbf{b}} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 1 + \tfrac{1}{3}\sum_{i=1}^{3} p_i, 0, 0, 0\right)'.$$

Moreover, the evaluator vector corresponding to the non-basic variables $(x_2, x_3, x_6, s_1, s_4, s_5)$ is

$$\hat{\mathbf{c}}'_N - \hat{\mathbf{c}}'_B\mathbf{B}^{-1}\mathbf{N} = (0, 0, 0, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3})$$

indicating that the solution $\hat{\mathbf{x}}_B$ is optimal. Proceeding as in the proof of Proposition 4, we calculate $\hat{\mathbf{x}}_B = \mathbf{B}^{-1}\hat{\mathbf{b}} - \mathbf{B}^{-1}\mathbf{N}\hat{\mathbf{x}}_N$ to express the basic variables as a function of the non-basic decision variables as follows:

$$x_1 = \frac{1}{3} - \frac{1}{3}(x_2 + x_3 + x_6) - \frac{1}{3}\left\{\frac{[2g(\mathbf{p}) - 3]a_1(\mathbf{x}) - [g(\mathbf{p}) - 3]a_2(\mathbf{x}) - g(\mathbf{p})a_3(\mathbf{x})}{3 - 3g(\mathbf{p}) + [g(\mathbf{p})]^2}\right\},$$

$$x_4 = \frac{1}{3} - \frac{1}{3}(x_2 + x_3 + x_6) - \frac{1}{3}\left\{\frac{[2g(\mathbf{p}) - 3]a_2(\mathbf{x}) - [g(\mathbf{p}) - 3]a_3(\mathbf{x}) - g(\mathbf{p})a_1(\mathbf{x})}{3 - 3g(\mathbf{p}) + [g(\mathbf{p})]^2}\right\},$$

$$x_5 = \frac{1}{3} - \frac{1}{3}(x_2 + x_3 + x_6) - \frac{1}{3}\left\{\frac{[2g(\mathbf{p}) - 3]a_3(\mathbf{x}) - [g(\mathbf{p}) - 3]a_1(\mathbf{x}) - g(\mathbf{p})a_2(\mathbf{x})}{3 - 3g(\mathbf{p}) + [g(\mathbf{p})]^2}\right\},$$

$$u_5 = 1 + \tfrac{1}{3}g(\mathbf{p}),$$

$$s_2 = \frac{a_4(\mathbf{x})[-x_3 g(\mathbf{p}) - x_6 g(\mathbf{p}) + 2x_2 g(\mathbf{p}) + 3(x_6 - x_2)]}{3 - 3g(\mathbf{p}) + [g(\mathbf{p})]^2},$$

$$s_3 = \frac{a_4(\mathbf{x})[-x_6 g(\mathbf{p}) - x_2 g(\mathbf{p}) + 2x_3 g(\mathbf{p}) + 3(x_2 - x_3)]}{3 - 3g(\mathbf{p}) + [g(\mathbf{p})]^2},$$

$$s_6 = \frac{a_4(\mathbf{x})[-x_2 g(\mathbf{p}) - x_3 g(\mathbf{p}) + 2x_6 g(\mathbf{p}) + 3(x_3 - x_6)]}{3 - 3g(\mathbf{p}) + [g(\mathbf{p})]^2},$$

where

$$\begin{aligned}
a_1(\mathbf{x}) &\equiv g_1(p_1)x_2 + g_3(p_3)x_3 + g_2(p_2)x_6,\\
a_2(\mathbf{x}) &\equiv g_3(p_3)x_2 + g_2(p_2)x_3 + g_1(p_1)x_6,\\
a_3(\mathbf{x}) &\equiv g_2(p_2)x_2 + g_1(p_1)x_3 + g_3(p_3)x_6,\\
a_4(\mathbf{x}) &\equiv 1 - g(\mathbf{p}) + p_1p_3 + p_1p_2 + p_2p_3.
\end{aligned}$$

First, note that $a_4(\mathbf{x}) > 0$ since it can be written as $a_4(\mathbf{x}) = (1-p_1)(1-p_2)(1-p_3) + p_1p_2p_3 > 0$. Next, the denominator of s_2, s_3, and s_6 is also positive since it can be simplified as $\frac{3}{4} + (\frac{3}{2} - \sum_{i=1}^{3} p_i)^2 > 0$. Thus,

$$\begin{aligned}
s_2 \geqslant 0 &\Leftrightarrow n_2(\mathbf{x}) \geqslant 0,\\
s_3 \geqslant 0 &\Leftrightarrow n_3(\mathbf{x}) \geqslant 0,\\
s_6 \geqslant 0 &\Leftrightarrow n_6(\mathbf{x}) \geqslant 0,
\end{aligned}$$

where

$$n_2(\mathbf{x}) = -x_3 g(\mathbf{p}) - x_6 g(\mathbf{p}) + 2x_2 g(\mathbf{p}) + 3(x_6 - x_2), \tag{28}$$

$$n_3(\mathbf{x}) = -x_6 g(\mathbf{p}) - x_2 g(\mathbf{p}) + 2x_3 g(\mathbf{p}) + 3(x_2 - x_3), \tag{29}$$

$$n_6(\mathbf{x}) = -x_2 g(\mathbf{p}) - x_3 g(\mathbf{p}) + 2x_6 g(\mathbf{p}) + 3(x_3 - x_6). \tag{30}$$

However, adding the $n_i(\mathbf{x})$'s in (28)–(30) we find $n_2(\mathbf{x}) + n_3(\mathbf{x}) + n_6(\mathbf{x}) = 0$. Since each $n_i(\mathbf{x})$ is non-negative but their sum is zero, this implies that $n_2(\mathbf{x}) = n_3(\mathbf{x}) = n_6(\mathbf{x}) = 0$. To determine the values of x_2, x_3, and x_6 we solve the system of three $\{n_2(\mathbf{x}) = 0, n_3(\mathbf{x}) = 0, n_6(\mathbf{x}) = 0\}$ in three unknowns $\{x_2, x_3, x_6\}$ which gives $x_2 = x_3 = x_6$ and

$$\begin{aligned}
x_1 &= \tfrac{1}{3} - \tfrac{1}{3}(x_2 + x_3 + x_6),\\
x_4 &= \tfrac{1}{3} - \tfrac{1}{3}(x_2 + x_3 + x_6),\\
x_5 &= \tfrac{1}{3} - \tfrac{1}{3}(x_2 + x_3 + x_6).
\end{aligned}$$

with $s_2 = s_3 = s_6 = 0$. As in the proof of Proposition 4, we find that $x_2 = x_3 = x_6 = \frac{1}{3}$ and thus $\mathbf{x}_b = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)'$ with the value of game $u_5 = 1 + \frac{1}{3}\sum_{i=1}^{3} p_i$. Maximizing and minimizing u_5, we find max $u_5 = 2^-$ and min $u_5 = 1^+$, respectively; thus $1 < u_5 < 2$. Finally, since $u_5 + v_5 = 3$, we obtain $v_5 = 2 - \frac{1}{3}\sum_{i=1}^{3} p_i$ as the value to Tianji. □

Appendix D. Proof of Proposition 6

To prove this proposition we use exactly the same lines of argument as those employed in proving Proposition 5 involving a linear program with non-numeric entries. That is, we start with the basis $\hat{\mathbf{x}}_B = (x_1, x_4, x_5, u_5, s_2, s_3, s_6)'$ corresponding to $\mathbf{x}_a$ and calculate its components as

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1}\hat{\mathbf{b}} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}, 0, 0, 0\right)'.$$

Next, we show that $\hat{\mathbf{x}}_B$ is one of many optimal solutions since the evaluator vector corresponding to the non-basic variables $\hat{\mathbf{x}}_N = (x_2, x_3, x_6, s_1, s_4, s_5)'$ is $\hat{\mathbf{c}}'_N - \hat{\mathbf{c}}'_B\mathbf{B}^{-1}\mathbf{N} = (0, 0, 0, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$. We then write $\hat{\mathbf{x}}_B$ in terms of the non-basic variables $\hat{\mathbf{x}}_N$ and show that $\mathbf{x}_b$ is the other optimal corner point.

Calculating $\hat{\mathbf{x}}_B = \mathbf{B}^{-1}\hat{\mathbf{b}} - \mathbf{B}^{-1}\mathbf{N}\hat{\mathbf{x}}_N$, we find

$$s_2 = \frac{r(\hat{\mathbf{p}})\gamma_2(\mathbf{x})}{m(\hat{\mathbf{p}})}, \quad s_3 = \frac{r(\hat{\mathbf{p}})\gamma_3(\mathbf{x})}{m(\hat{\mathbf{p}})}, \quad s_6 = \frac{r(\hat{\mathbf{p}})\gamma_6(\mathbf{x})}{m(\hat{\mathbf{p}})}, \tag{31}$$

where

$$\gamma_2(\mathbf{x}) = w_3(\hat{\mathbf{p}})x_2 + w_2(\hat{\mathbf{p}})x_3 + w_1(\hat{\mathbf{p}})x_6,$$
$$\gamma_3(\mathbf{x}) = w_1(\hat{\mathbf{p}})x_2 + w_3(\hat{\mathbf{p}})x_3 + w_2(\hat{\mathbf{p}})x_6,$$
$$\gamma_6(\mathbf{x}) = w_2(\hat{\mathbf{p}})x_2 + w_1(\hat{\mathbf{p}})x_3 + w_3(\hat{\mathbf{p}})x_6,$$

and

$$w_1(\hat{\mathbf{p}}) = 2(p_{12} + p_{23} + p_{31}) - (p_{11} + p_{22} + p_{33} + p_{13} + p_{21} + p_{32}),$$
$$w_2(\hat{\mathbf{p}}) = 2(p_{13} + p_{21} + p_{32}) - (p_{11} + p_{22} + p_{33} + p_{31} + p_{12} + p_{23}),$$
$$w_3(\hat{\mathbf{p}}) = 2(p_{11} + p_{22} + p_{33}) - (p_{12} + p_{13} + p_{21} + p_{23} + p_{31} + p_{32}).$$

with the property that $\gamma_2(\mathbf{x}) + \gamma_3(\mathbf{x}) + \gamma_6(\mathbf{x}) = (x_2 + x_3 + x_6)\sum_{i=1}^{3} w_i(\hat{\mathbf{p}})$ and $w_1(\hat{\mathbf{p}}) + w_2(\hat{\mathbf{p}}) + w_3(\hat{\mathbf{p}}) = 0$ implying $\gamma_2(\mathbf{x}) + \gamma_3(\mathbf{x}) + \gamma_6(\mathbf{x}) = 0$. In these expressions the term $r(\hat{\mathbf{p}})$ in the numerators of s_i's in (31) is a quadratic form given as $r(\hat{\mathbf{p}}) = \hat{\mathbf{p}}'\mathbf{Q}\hat{\mathbf{p}}$, where

$$\mathbf{Q} = \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_2 & \mathbf{0} & \mathbf{Q}_1 \\ \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{0} \end{bmatrix}_{9\times 9},$$

with

$$\mathbf{Q}_1 \equiv \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}_{3\times 3}, \quad \mathbf{Q}_2 = -\mathbf{Q}_1,$$

and $\mathbf{0}$ is a 3×3 matrix of 0's. The term $m(\hat{\mathbf{p}})$ appearing in the denominators in (31) is also a quadratic form $m(\hat{\mathbf{p}}) = \hat{\mathbf{p}}'\hat{\mathbf{Q}}\hat{\mathbf{p}}$ where $\hat{\mathbf{Q}} \equiv (\hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2, \hat{\mathbf{Q}}_3, \hat{\mathbf{Q}}_3, \hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2, \hat{\mathbf{Q}}_2, \hat{\mathbf{Q}}_3, \hat{\mathbf{Q}}_1)$ is a 9×9 matrix with

$$\hat{\mathbf{Q}}_1 \equiv (1, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 1)',$$
$$\hat{\mathbf{Q}}_2 \equiv (-\tfrac{1}{2}, 1, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 1, 1, -\tfrac{1}{2}, -\tfrac{1}{2})',$$
$$\hat{\mathbf{Q}}_3 \equiv (-\tfrac{1}{2}, -\tfrac{1}{2}, 1, 1, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 1, -\tfrac{1}{2})'.$$

To assure that $s_2 \geqslant 0$, $s_3 \geqslant 0$, and $s_6 \geqslant 0$, the terms $\gamma_2(\mathbf{x})$, $\gamma_3(\mathbf{x})$ and $\gamma_6(\mathbf{x})$ must have the same sign. But since their sum equals zero, this implies that $\gamma_2(\mathbf{x}) = \gamma_3(\mathbf{x}) = \gamma_6(\mathbf{x}) = 0$. To determine the values of x_2, x_3 and x_6 we solve the system of three $\{\gamma_2(\mathbf{x})=0, \gamma_3(\mathbf{x})=0, \gamma_6(\mathbf{x})=0\}$ in three unknowns $\{x_2, x_3, x_6\}$ which gives $x_2 = x_3 = x_6 = \bar{x}$ and $x_1 = \frac{1}{3} - \bar{x}$, $x_4 = \frac{1}{3} - \bar{x}$, and $x_5 = \frac{1}{3} - \bar{x}$. As in the proof of Proposition 4, we find that $x_2 = x_3 = x_6 = \frac{1}{3}$ and thus $\mathbf{x}_b = (0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3})'$with the value of game $u_6 = \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}$. Maximizing and minimizing u_6, we find max $u_6 = 3^-$ and min $u_6 = 0^+$, respectively; thus $0 < u_6 < 3$. Finally, since $u_6 + v_6 = 3$, we obtain $v_6 = 3 - \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}$ as the value to Tianji. □

References

[1] Sun-Tzu. The art of war. Oxford: Clarendon Press; 1963. Translation of Sun-Tzu's 6th century B.C. book by Samuel B. Griffith.

[2] Tang Z-C. Principles of conflict: recompilation and new english translation with annotations on Sun Zi's art of war. San Rafael, CA: T. C. Press; 1969.

[3] Shubik M. Game theory in the social sciences. Cambridge, MA: The MIT Press; 1982.

[4] Straffin PD. Game theory and strategy. Washington, DC: The Mathematical Association of America; 1993.

[5] Gerchak Y. Operations research in sports. In: Pollock SM, Rothkopf MH, Barnett A, editors. Operations research and the public sector. Handbooks in operations research and management science. Amsterdam: North-Holland; 1994. p. 507–27.

[6] Ladany SP, Machol RE, editors. Optimal strategies in sports. Studies in Management Science and Systems. Amsterdam: North-Holland Publishing Company; 1977.

[7] Machol RE, Ladany SP, Morrison DG, editors. Management science in sports. North-Holland/TIMS Studies in the Management Sciences. Amsterdam: North-Holland; 1976.

[8] Dresher M. Games of strategy: theory and applications. Englewood Cliffs, NJ: Prentice-Hall, Inc.; 1961.

[9] Shannon C. A mathematical theory of communication. Bell System Technical Journal 1948;27:379–423.

[10] Ash RB. Information theory. New York: Interscience Publishers; 1965.

[11] Schrage L. Optimization modeling with LINGO. Chicago: LINDO Systems Inc.; 2003.

[12] Char BW. Maple 8 learning guide. Waterloo, Canada: Waterloo Maple; 2002.

[13] Parlar M. Interactive operations research with Maple: methods and models. Boston: Birkhäuser; 2000.

[14] Dantzig GB. Linear programming and extensions. Princeton, NJ: Princeton University Press; 1963.

[15] Wang J. The theory of games. Oxford: Clarendon Press; 1988.

[16] Zionts S. Linear and integer programming. Englewood Cliffs, NJ: Prentice-Hall; 1974.

[17] Schrage L. Optimization modeling with LINDO, 5th ed. Pacific Grove, CA: Brooks/Cole; 1997.

[18] Gass S. Linear programming: methods and applications. New York: McGraw-Hill, Inc.; 1969.