Recent Developments in Dynamic Advertising Research[1]

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Abstract

A variety of continuous-time differential functions have been developed to investigate dynamic advertising problems in business and economics fields. Since major dynamic models appearing before 1995 have been reviewed by a few survey papers, we provide a comprehensive review of the dynamic advertising models published after 1995, which are classified into six categories: (i) Nerlove-Arrow model and its extensions, (ii) Vidale-Wolfe model and its extensions, (iii) Lanchester model and its extensions, (iv) the diffusion models, (v) dynamic advertising-competition models with other attributes, and (vi) empirical studies for dynamic advertising problems. For each category, we first briefly summarize major relevant before-1995 models, and then discuss major after-1995 models in details. We find that the dynamic models reviewed in this paper have been extensively used to analyze various advertising problems in the monopoly, duopoly, oligopoly, and supply chain systems. Our review reveals that the diffusion models have not been used to analyze advertising problems in supply chain operations, which may be a research direction in the future. Moreover, we learn from our review that very few publications regarding dynamic advertising problems have considered the supply chain competition. We also find that very few researchers have used the diffusion model to investigate the dynamic advertising problems with product quality as a decision variable; and, the pricing decision has not been incorporated into any extant Lanchester model. The paper ends with a summary of our review and suggestions on possible research directions in the future.

Key words: Advertising; dynamics; Nerlove-Arrow; Vidale-Wolfe; Lanchester; diffusion; empirical studies

1 Introduction

Advertising is a common communication form that can be used to induce consumers to take some actions with respect to products or services. To drive consumer behavior regarding a commercial offering, a firm may pay to distribute advertising messages via various traditional mass media such as newspaper, magazines, television commercial, radio advertisement, outdoor advertising, or direct mail; or via some new media including, e.g., websites, text messages, etc. As a main tool for "persuading" consumers to buy, advertising possesses two major functions, according to a discussion by Comanor and Wilson [24]. The first major function is about advertising's "channel" role, with which advertising can provide valuable information to consumers, and enable them to make rational choices by reducing informational product differentiation. The second major function is concerned with advertising's "differentiation" role, with which advertising persuades consumers by means of intangible and/or psychic differentiators and creates differentiation among products.

Since advertising significantly impacts consumers' purchasing behaviors, each firm should spend its effort (by making an advertising investment) to "promote" its product or service to consumers via advertising media; but, this should be subject to the firm's limited budget. Therefore, for each firm, a proper decision on the advertising effort should be made to boost the sales and improve the firm's profit. Because, in reality, the impact of advertising on demand usually varies over time, an important research question arises as follows: how does a firm dynamically allocate its limited expenditure over time to boost its sales or maximize its profit? To address this problem, one may need to construct an appropriate dynamic advertising model to characterize the time-dependent sales. This has motivated researchers to build a variety of models for diverse problems in economics, marketing, and operations management fields.

After the initial excitement resulting from the applications of dynamic models to advertising problems in the early 1960s, a great number of dynamic models in discrete- and continuous-time differential forms have appeared in the past five decades to analyze firms' advertising policies over time. The dynamic models usually involve an important dimension of time to reflect the time-dependence of advertising decisions. Even though there exist some discrete-time models (see the reviews by, e.g., Erickson [38] and Sethi [138]), the continuous-time models are still significantly more common in extant publications. Accordingly, we focus our review on the continuous-time models, which are usually used for optimal control problems or for differential games. Specifically, if a monopoly firm determines its optimal advertising level in the absence of competition, then we should construct an optimal control model with a continuous-time differential function; otherwise, if two or more firms determine their advertising policies to compete in a market, then we should formulate a continuous-time differential game model. For the applications of optimal control theory in management science and differential game models in marketing, see Sethi and Thompson [141] and Jørgensen and Zaccour [78], respectively. In the dynamic models, the critical state variables, such as sales, market share, brand goodwill, etc., are commonly assumed to change with respect to time, as described by differential equations. Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138] provided three comprehensive reviews for the dynamic advertising models published prior to the year 1995; Erickson [38] discussed the dynamic models from the perspective of differential games in the advertising competition, and Feichtinger, Hartl, and Sethi [44] and Sethi [138] performed their reviews from the optimal control perspective.

Since 1995, the interest in dynamic advertising problems has not waned but instead continued to grow as a number of researchers have been applying the optimal control theory and the differential game theory to investigate various advertising problems in new business or economics settings. For example, starting from the mid-1990s, supply chain analysis has been an important research direction in the operations management area, possibly because of the rapid development of information technology. A number of researchers have accordingly investigated dynamic advertising policies in supply chain operations, mostly concerning a two-echelon supply chain where a manufacturer decides whether or not to share the advertising expense incurred by a retailer in promoting the manufacturer's product via advertising media. The relevant publications include Chintagunta and Jain [21], He et al. [66], Jørgensen et al. [71], etc.

Some dynamic advertising-related publications appearing after 1995 are associated with the advertising competition between two, or among three or more brands. For instance, a retailer selling several brands should decide to properly allocate its advertising cost among the brands. Such an allocation problem has been examined by, e.g., Erickson [40] and Martín-Herrán et al. [99]. As another example, a manufacturer and a retailer in a supply chain need to determine their optimal advertising policies for the national brand and the store brand, respectively; for relevant publications, see Amrouche et al. [1] and [2]. In recent years, some researchers have developed new dynamic models to analyze firms' advertising policies for new product introduction. For example, Mosca and Viscolani [109] used a modified Nerlove-Arrow model to determine an optimal goodwill path for the introduction of a new product; Sethi and He [140] investigated the optimal advertising and pricing decisions for a firm who introduces a new product to a market.

According to our above discussion, we find that, after 1995, a number of continuous-time differential functions have been developed to investigate new dynamic advertising problems. It thus behooves us to perform an updated review on the dynamic advertising models emerging after 1995, which reflects the "recent" developments of dynamic advertising study. In fact, we review more than 160 relevant publications in this paper, describing the after-1995 dynamic advertising-related research activities. Noting that the dynamic models published before 1995 have been reviewed by Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138], we shall not discuss those before-1995 publications in our review; but for the sake of completeness, we list major before-1995 dynamic advertising models.

The dynamic advertising literature contains a large number of continuous-time dynamic optimization models and differential-game models in different mathematical forms for diverse advertising problems. To facilitate our review in an orderly manner, we classify our reviewed models into six groups according to their mathematical formulations, which include (i) Nerlove-Arrow model and its extensions (which are reviewed in Section 2); (ii) Vidale-Wolfe model and its extensions (which are reviewed in Section 3); (iii) Lanchester model and its extensions (which are reviewed in Section 4); (iv) the diffusion models (which are reviewed in Section 5); (v) dynamic advertising-competition models with other attributes such as sticky price and reservation price, which include those major models that do not fall into the first four categories, as reviewed in Section 6; and (vi) empirical studies for dynamic advertising problems. For each category, we review the relevant models with an emphasis on their specific mechanisms leading to the advertising dynamics. This paper ends with a summary of our review and suggestions on the future dynamic advertising-related research in Section 8.

2 Nerlove-Arrow Model and Extensions

Nerlove and Arrow [117] developed the classical Nerlove-Arrow (N-A) model to describe the time-dependent demand in a general function form. Specifically, using the N-A model, we can calculate the demand for a firm's product at time t as $S(t) = f(A(t), p(t), Z(t))$, where $A(t)$ denotes the goodwill stock summarizing the impacts of the firm's current and past advertising outlays on the demand, and evolves over time according to the following dynamic equation:

$$\dot{A}(t) \equiv dA(t)/dt = u(t) - \delta A(t), \quad \text{with } A(0) = A_0, \tag{1}$$

in which $u(t)$ is the firm's advertising outlay at time t and δ is a constant proportional rate at which depreciation occurs. Moreover, in the N-A model $S(t)$, $p(t)$ is the price charged for the product at time t, and $Z(t)$ is a variable representing other factors (except for the goodwill stock and the price) that influence the demand but are not under the firm's control—e.g., consumer incomes, population, and the prices of substitute and complementary products.

A number of scholars in the economics, marketing, and operations management areas have extended the above classic N-A model to construct a variety of continuous-time demand functions. The relevant models published prior to 1995 have been reviewed by Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138]; for a brief summary of those reviewed models, see online Table A in the online appendix. We note that, in the models appearing before 1995, a firm's capital stocks such as goodwill and reputation are dependent on the firm's advertising effort/outlay. Among the models in Table A, the advertising dynamics functions proposed by Rao [130], Rishel [132], and Tapiero ([151] and [152]) are regarded as the stochastic generalizations of the N-A model. Rao [130] and Rishel [132] developed two stochastic models using the birth-death process, and Tapiero ([151] and [152]) built two models using the Wiener process. In addition, Chintagunta [20] and Fershtman et al. [49] developed models to analyze the advertising competition between two firms; Fershtman [48], Pauwels [124], Rao [130], and Rishel [132] constructed their models to characterize the advertising competition in an oligopoly setting.

Even though some papers ([38], [44], and [138]) have reviewed the N-A model and extensions that were published before 1995, we find that, after 1995, a number of scholars continued to extend the N-A model to develop some new models for investigating various business- and economics-related problems. It is thus worthwhile to review the "recent" N-A extensions (that were built after 1995), which are classified into three categories: the extended N-A models in the monopoly setting; the extended N-A models in the duopoly and oligopoly settings; and the extended N-A models in supply chain operations.

2.1 The Extended Nerlove-Arrow Models for Dynamic Advertising Problems in the Monopoly Setting

We now review the after-1995 extensions of the N-A model, which characterize the advertising effort-dependent demand faced by a single firm with no competition. According to the topics concerned by the relevant publications, we categorize these models into four groups: new-product introduction models; advertising-pricing models; stochastic advertising models; and the N-A extensions with other attributes.

2.1.1 The New-Product Introduction Models

Mosca and Viscolani [109] determined an optimal goodwill path maximizing a foreseen profit for the introduction of a new product in a market. Generalizing the original N-A model in (1), the authors characterized the goodwill stock in the following continuous-time differential equation form: $\dot{A}(t) = f(u(t)) - \delta A(t)$, where $f : [0, +\infty) \to [0, +\infty)$ is a strictly increasing and concave function explaining how the advertising effort produces new goodwill. Note that, prior to Mosca and Viscolani [109], Buratto and Viscolani ([13] and [14]) had considered a special case of the above model by specifying the function $f(u(t))$ as $f(u(t)) = [u(t)]^\alpha$, where $\alpha \in (0,1]$ is a constant. Mosca and Viscolani [109] obtained an optimal solution maximizing a firm's foreseen profit that involves several specific cost function forms such as the linear-quadratic cost function and the N-A type cost rate function.

Buratto et al. ([11] and [12]) investigated a firm's advertising policy in a market with a finite number of segments consisting of a consumer population. In [11] Buratto et al. considered a market with the size M, where each consumer segment is specified by a segmentation attribute $a \in \Omega$ with Ω denoting a set of the attributes. For the time interval $[0,T]$ ($T < +\infty$) devoted to advertising for the introduction of a new product, Buratto et al. [11] extended the classical N-A model to the following evolution equation of the goodwill:

$$\dot{A}(t,a) \equiv \partial A(t,a)/\partial t = u(t,a) - \delta(a)A(t,a) \text{ with } A(0,a) = \alpha(a) \geq 0, \text{ for } a \in \Omega, \tag{2}$$

where, for the segment with the attribute a, $A(t,a)$ represents the stock of goodwill of the product at time $t \in [0,T]$, $\delta(a) > 0$ means the goodwill depreciation rate for the members in the segment, and $u(t,a)$ is the effective advertising intensity at time t directed to the segment.

In [12], Buratto et al. extended the model in (2) to analyze a case in which several advertising channels are available with different diffusion spectra and efficiencies. First, for a single advertising channel, the authors replaced the term $u(t,a)$ in the goodwill evolution function (2) with $\gamma(a)\varphi(u(t))$, where $\gamma(a)$ denotes the channel spectrum such that $\sum_{a\in\Omega}\gamma(a) = 1$ and $\varphi(u(t)) \geq 0$ is the channel's effective advertising level reflecting the fact that the effect of advertising intensities on the goodwill variable may vary with the advertising channel. Each advertising channel is characterized by $(\gamma(a), \varphi(u(t)))$. Then, assuming that $\varphi(u(t)) = s_i\sqrt{u_i(t)}$, Buratto et al. examined a special case in which a firm needs to choose one from n available advertising channels to direct the pre-launch campaign for a new product.

Different from Buratto et al. ([11] and [12]), who used the ordinary differential equation (ODE) to capture the effect of advertising on the goodwill, Marinelli and Savin [97] proposed a partial differential equation (PDE) to characterize the goodwill dynamics on both space and time coordinates; that is,

$$\partial A(t,\hat{a})/\partial t = -\delta A(t,\hat{a}) + \triangle_{\hat{a}} A(t,\hat{a}) + b(\hat{a})u(t,\hat{a}), \tag{3}$$

where $A : [0,T] \times \Xi \to \mathbb{R}$ with $\Xi \subset \mathbb{R}$ is the goodwill "density," $\hat{a}$ is a spatial coordinate, $u(t,\hat{a})$ is the rate of advertising effort, and $b(\hat{a})$ is the coefficient of advertising effectiveness. Note that the second term on the right-hand side of (3) reflects the effect of the spatial diffusion of goodwill.

2.1.2 The Advertising-Pricing Models

We now review two major N-A extensions where, for a firm, the brand goodwill (or the sales) function is dependent on both the firm's advertising effort and selling price. Such extensions include (i) the quality-signaling model, and (ii) the marketing and operations interface model.

The Quality-Signaling Model Motivated by the fact that customers may assess experience goods' qualities only after consuming those products, some scholars investigated the dynamic advertising problems in which the product quality may influence consumers' choices. For example, Spremann [148] assumed that a seller's reputation—as a signal of the quality of the seller's product—depends on the quality-price ratio as perceived by former customers and the quantity already used by the customers; for Spremann's model, see Table A in the online appendix. Spremann [148] showed that the prices above the reference price depress the sales rate, which in turn slow down the decline in reputation. To avoid this, Feichtinger et al. [45] considered a two-stage purchase decisions. Similar to Spremann [148], Conrad [25] addressed an asymmetric information case in which the quality of a good can be perceived by consumers only after their purchases, assuming that the firm determines the optimal quantity, quality, and advertising level to maximize its profit. For the before-1995 (published before 1995) models regarding the product quality, see the literature review conducted by Feichtinger, Hartl, and Sethi [44].

Differing from the before-1995 models on quality signaling, Fruchter [54] investigated the roles of price and advertising level that serve as the quality indicators in a dynamic framework, assuming that the price is used both as a monetary constraint and as a quality signal, but the advertising expenditure is used only as a signaling device and thus a purely dissipative expense. Fruchter modified the classical N-A model in (1) to $\dot{A}(t) = kp(t) + \rho u(t) - \delta A(t)$ with $A(0) = A_0$, where $A(t)$ is the perceived quality of a particular brand at time t and can be regarded as the accumulated goodwill of the brand at time t; the constants k and ρ represent the direct price and advertising effects on the brand's perceived quality, respectively; and the constant δ means the rate of perceived quality loss resulting from consumers' "fatigue" on the brand. The rate of sales $S(p(t), A(t))$ is decreasing in the price $p(t)$ but increasing in the brand's perceived quality (i.e., goodwill) $A(t)$.

The Marketing and Operations Interface Model Erickson [42] considered the joint pricing and advertising decisions for a firm involving two functional areas—marketing and operations, which usually aim at different objectives. Specifically, the former seeks to enhance demand and the latter intends to minimize the operating (e.g., manufacturing) cost. Erickson adopted the classic N-A model in (1) to describe the goodwill evolution, and developed the price- and advertising-dependent sales function as $S(t) = \alpha - \beta p(t) + \gamma A(t)$, where α is a constant and β is a measure of the negative effect of price on the sales. Assuming that the backlogging is allowed, Erickson characterized the dynamic backlog variable as $\dot{B}(t) = S(t) - \kappa(t)$, where $\kappa(t)$ denotes the dynamic production rate. In the firm, the marketing department determines $p(t)$ and $u(t)$—which is a term of the goodwill $A(t)$ as given in (1), and the operations department decides on $\kappa(t)$. Erickson derived the feedback Nash equilibrium and the feedback Stackelberg equilibrium with the marketing department as the decision leader.

2.1.3 The Stochastic Advertising Models

Feichtinger, Hartl, and Sethi [44] and Sethi [138] have extensively reviewed the before-1995 stochastic extensions of the classical N-A model, which mainly include the random walk model (see, e.g., Rishel [132], Tapiero [151] and [152]) and the lag models and their estimation (see, e.g., Rao [130]).

Following the above avenues, Raman [129] investigated a boundary value problem in the stochastic optimal control of advertising, where the salvage value at the end of the planning horizon may affect a firm's advertising decision. Similar to Rao [130], Raman postulated a stochastic differential equation for the goodwill process as $dA(t) = [\beta u(t) - \delta A(t)]\, dt + \sigma d\omega(t)$ with $A(0) = A_0$, where σ is the infinitesimal standard deviation and $\omega(t)$ is a standard Brownian motion process. Raman derived the optimal advertising policy and the optimal length of the planning horizon.

Similar to Raman [129], Marinelli [96] determined the optimal advertising policy for a firm who desires to maximize the utility of goodwill at the launch time and minimize the disutility of a stream of advertising costs that extend until the launch time. Since the emergence of randomness in the dynamics of goodwill is natural, Marinelli introduced some stochastic extensions to the classic N-A model and developed the goodwill dynamics as $dA(t) = [u(t) - \rho A(t)]dt + \sigma(A(t), u(t))d\omega(t)$ with $A(0) = A_0$, where ρ is a positive constant; $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous; and, the control process $u(t)$ is a measurable, adapted variable denoting the firm's advertising effort, taking values in a closed convex subset U of $[0, +\infty)$.

Gozzi et al. [62] considered a class of dynamic advertising problems for a firm in the presence of carryover and distributed forgetting effects, assuming that the goodwill depends on its past values as well as previous advertising levels. The firm chooses its advertising effort to influence the product goodwill that evolves according to the following stochastic differential equation,

$$dA(t) = \left[\lambda_0 A(t) + \int_{-\gamma}^{0} \lambda_1(\theta)A(t+\theta)d\theta + \hat{\lambda}_0 u(t) + \int_{-\gamma}^{0} \hat{\lambda}_1(\theta)u(t+\theta)d\theta\right] dt + \sigma d\omega(t),\ t_0 \leq t \leq T, \quad (4)$$

with $A(t_0) = A_0$, $A(t_0 + \theta) = A_1(\theta) \geq 0$, $u(t_0 + \theta) = \delta(\theta)$, and $\theta \in [-\gamma, 0]$. Gozzi et al. assumed that the advertising spending rate $u(t)$ is constrained to remain in the set U, which is the space of $\mathbb{F}$-adapted process taking values in a compact interval $\mathcal{U} \subseteq \mathbb{R}_+$. Moreover, they assumed the following conditions: (i) $\lambda_0 \leq 0$ and $\lambda_1(\cdot) \in L^2([-\gamma, 0], \mathbb{R})$; (ii) $\hat{\lambda}_0 \geq 0$ and $\hat{\lambda}_1(\cdot) \in L^2([-\gamma, 0], \mathbb{R})$; and (iii) $\delta(\cdot) \geq 0$. Here, λ_0 and $\lambda_1(\cdot)$ describe the process of goodwill deterioration when the advertising terminates, and $\hat{\lambda}_0$ and $\hat{\lambda}_1(\cdot)$ provide the characterization of the effects of the current and past advertising rates on the goodwill level. The values of A_0, $A_1(\cdot)$, and $\delta(\cdot)$ reflect the initial goodwill and advertising trajectories.

2.1.4 The Extended Nerlove-Arrow Models with Other Attributes

We now review the after-1995 N-A extensions that addressed issues other than new product introduction and advertising-pricing decisions. These extensions are mainly concerned with (i) the optimal advertising investment in a spatial monopoly (Lambertini [88]), (ii) the advertising coordination in a fashion licensing contract (Buratto and Zaccour [15]), (iii) the advertising decision in a franchise system (Sigué and Chintagunta [144]), (iv) the optimal dynamic advertising with an adverse exogenous effect on brand goodwill (Grosset and Viscolani [63]), and (v) the wearout effects of different advertising themes (Bass et al. [5]).

Advertising Investment in a Spatial Monopoly Lambertini [88] considered the advertising investment in a spatial monopoly and contrasted the socially optimal behavior of a benevolent against that of a profit-seeking monopolist. As an extension of the classic N-A model in (1), Lambertini's advertising dynamics was given as $\dot{p}_r(t) = \delta_1\sqrt{u(t)} - \delta_2 p_r(t)$, for $\delta_1 > 0$, where $p_r(t)$ is consumers' reservation price (that is equivalent to the willingness to pay for the product) and δ_2 is the constant depreciation rate affecting consumers' willingness to pay. A social planner aims at maximizing the social welfare $\int_0^\infty e^{-\gamma t}[S_W(t) - \rho u(t)]dt$, where $\rho u(t)$ means the advertising cost and $S_W(t)$ is the instantaneous social welfare calculated as the sum of the planner's instantaneous revenue $R(t)$ and the instantaneous consumer surplus $C_S(t)$, i.e., $S_W(t) = R(t) + C_S(t)$. A monopoly firm maximizes its discounted profit $\int_0^\infty e^{-\gamma t}[R(t) - \rho u(t)]dt$. Specifying the functions $R(t)$ and $C_S(t)$, Lambertini derived the optimal advertising decisions when either the social planner or the monopoly firm is the decision maker.

Advertising Coordination in a Fashion Licensing Contract Licensing is the process of leasing a licensor's legally protected entity (e.g., brand, name, and logo) associated with a product or product line based on an agreement between the licensor and the licensee. Buratto and Zaccour

[15] investigated the cooperative and non-cooperative advertising strategies between a licensor and a licensee over the planning horizon $[0, T]$. They considered two types of licensing agreements. In the first type, the licensor of a famous fashion brand grants to a licensee the rights to produce and market its second-line brand based on monetary compensation. The evolution of the front-line brand goodwill was described as $\dot{A}_L(t) = \eta_L u_L(t) - \delta A_L(t)$ with $A_L(0) = \bar{A}_L$, where $\delta > 0$ is the decay rate, $\eta_L > 0$ is the efficiency of the licensor's advertising, and $\bar{A}_L$ represents the initial goodwill value. The evolution dynamics of the second-line brand goodwill was given as $\dot{A}_l(t) = \eta_l u_l(t) + \beta_L A_L(t) - \delta A_l(t)$ with $A_l(0) = \bar{A}_l$, where $\bar{A}_l > 0$ is the initial goodwill value, $\eta_l > 0$ is the efficiency of the second-line brand advertising, and $\beta_L > 0$ is the spillover from the front-line goodwill to the reputation of the second-line. Moreover, Buratto and Zaccour assumed that the rate of sales revenue, $\dot{S}(t)$, depends linearly on the goodwill stocks $A_L(t)$ and $A_l(t)$; that is, $\dot{S}(t) = \alpha_L A_L(t) + \alpha_l A_l(t)$ with $S(0) = 0$, where α_L and α_l are the scaling parameters transforming the corresponding goodwill stock into sales.

In the second type of licensing agreement, the licensee obtains the rights to produce and market the same brand in a complementary business. For this case, only one brand was involved, and thus, one goodwill stock evolves according to the dynamics as $\dot{A}(t) = \gamma_L A_L(t) + \gamma_l A_l(t) - \delta A(t)$ with $A(0) = \bar{A}$, where γ_L and γ_l represent the advertising efficiencies for the licensor and the licensee, respectively. The cumulative sales $S_C(t)$ depend on the goodwill as indicated by the differential equation that $\dot{S}_C(t) = \alpha A(t)$ with $S_C(0) = 0$.

For the above two types of licensing agreements, Buratto and Zaccour also derived the Stackelberg equilibria for the leader-follower game where the licensor is the leader and the licensee is the follower. The licensor and the licensee maximize their profits to determine the optimal advertising efforts, $u_L(t)$ and $u_l(t)$, respectively. It was shown that, using an incentive strategy that depends on the licensee's advertising effort, the licensor can coordinate the decentralized system and induce the jointly optimal solutions.

Optimal Advertising with an Adverse Exogenous Effect on Brand Goodwill Grosset and Viscolani [63] investigated a firm's optimal advertising decision in a homogenous market with a constant exogenous interference that plays a negative effect in an additive way. The goodwill could be negative, and a zero demand is associated with negative goodwill values. Extending the classic N-A model in (1), Grosset and Viscolani proposed their goodwill dynamics as $\dot{A}(t) = \gamma_1 u(t) - \gamma_2 - \delta A(t)$ with $A(0) = A_0 > 0$, where γ_1 is the advertising efficiency, γ_2 is the constant exogenous interference, and δ is the goodwill depreciation rate. The term $\gamma_1 u(t) - \gamma_2$ represents the effective advertising intensity which can be negative or positive and may lead to negative goodwill values. Accordingly, Grosset and Viscolani assumed a piecewise linear function for the goodwill-dependent demand as $S(A(t)) = \beta \max\{0, A(t)\}$, where $\beta > 0$ is the marginal demand with respect to the goodwill when it is positive.

Wearout Effects of Different Advertising Themes In practice, a firm may use different themes of advertising—e.g., price advertisements, product advertisements—and within each theme, the firm may consider different versions of an advertisement. To examine the optimal advertising policy for such a problem, Bass et al. [5] developed a model to jointly analyze the effects of wearout and forgetting in an advertising campaign that employs $k \geq 2$ different advertising themes. They quantified the differential wearout effects across different themes of advertising and examined the interaction effects among different themes using a Bayesian dynamic linear model. Bass et al. extended the classic N-A model in (1) and the model by Naik et al. [113] to the following advertising dynamics:

$$\begin{aligned}\dot{A}(t) &= \sum_{i=1}^{k} \psi_i(t) \left[g_i(u_i(t)) + \lambda_i \sum_{j=1, j\neq i}^{k} h(u_i(t), u_j(t)) \right] - \delta A(t), \\ \dot{\psi}_i(t) &= -a(u_i(t))\psi_i(t) + \delta[1 - I(u_i(t))][1 - \psi_i(t)],\end{aligned} \tag{5}$$

where $\psi_i(t)$ is the effectiveness of advertising theme i ($i = 1, \ldots, k$), $g_i(u_i(t))$ is a function of the advertising expenditure for theme i, and $a(u_i(t)) \equiv c_i + w_i u_i(t)$ with c_i as the copy wearout parameter and w_i as the repetition wearout parameter—which both reflect the advertising effect. Moreover, in (5), the term $\lambda_i \sum_{j=1, j\neq i}^{k} h(u_i(t), u_j(t))$ captures the interaction effects among theme i and other themes; and $I(u_i(t))$ is an indicator function that takes the value of "1" when there is advertising expenditure at time t.

The Infinite-Horizon Optimal Advertising for Durable Goods Weber [163] considered the optimal advertising decision for durable goods in a market over an infinite time horizon. Incorporating the effect of replacement sales due to obsolescence after a characteristic product lifetime of $1/\beta$, Weber extended the N-A model in (1) to characterize the accumulation of the advertising effect $A(t)$ as $\dot{A}(t) = [u(t)]^k - \alpha A(t)$ where $\alpha > 0$ and $k \in (0, 1)$ represents the impact of decreasing returns on the advertising effort. Weber also extended the Vidale-Wolfe (V-W) model [159] to capture the evolution of the fraction of consumers who own the products by using $\dot{x}(t) = [1 - x(t)]A(t) - \beta x(t)$, where $[1 - x(t)]A(t)$ represents the sales of new products and $\beta x(t)$ ($\beta > 0$) means the obsolescence of old products. Moreover, Weber derived the conditions under which the optimal solution exists.

2.2 The Extended Nerlove-Arrow Models for Dynamic Advertising Problems with Horizontal Competition

We review recent (published after 1995) N-A extensions that were developed to analyze dynamic advertising-related problems in the duopoly or oligopoly setting. For the differential advertising models involving the horizontal competition between two, or among three or more firms, see Erickson ([35] and [38]).

2.2.1 The Advertising-Quality Model

Nair and Narasimhan [116] used a differential game approach to investigate an advertising problem in a duopoly setting with the goodwill formation depending on advertising and the product quality. Each firm determines its price, advertising effort, and the investment on quality to maximize its discounted profit. Nair and Narasimhan characterized the demand faced by firm i at time $t \in [0, \infty)$ as $S_i(t) = \max\{0, [\alpha - \beta p_i(t)][g_1 A_i(t) - g_2 [A_i(t)]^2/2]\}$, where α, β, g_1, and g_2 are positive constants, and $p_i(t)$ and $A_i(t)$ are firm i's selling price and goodwill at time t, respectively. The goodwill dynamics for firm i was given as $\dot{A}_i(t) = q_i(t) - \theta_q q_j(t) + u_i(t) - \theta_q u_j(t) - \delta A_i(t)$ with $A_i(0) = \bar{A}_i$ (i, $j = 1, 2$ and $i \neq j$), where $q_i(t)$ and $u_i(t)$ denote the quality and advertising investments made by firm i at time t, respectively, and $\delta > 0$ is the depreciation parameter.

2.2.2 The Advertising-Pricing Models

Bertuzzi and Lambertini [9] investigated a differential game where two firms compete in a market for horizontally-differentiated products by determining their locations, prices, and advertising investments. Similar to (but different from) Hotelling model [68], consumers in the market—where the number of potential consumers at time $t \in [0, \infty)$ is $M(t)$—are uniformly distributed along the unit interval $[0, 1]$. At time t, firm i ($i = 1, 2$) chooses its location $y_i(t) \in [0, 1]$. When a consumer residing at $l \in [0, 1]$ buys one unit of firm i's product, the consumer can obtain the surplus $U = s - p_i(t) - g(y_i(t) - l) \geq 0$, where s is the consumer's reservation price, $p_i(t)$ is firm i's selling price, and $g(\cdot)$ is the transportation cost function.

Assuming that $g(y_i(t) - l) = k \times |y_i(t) - l|$, Bertuzzi and Lambertini [9] found that the consumer residing at $\bar{l}(t) \equiv [p_2(t) - p_1(t) + k(y_1(t) + y_2(t))]/(2k)$ is indifferent between products 1 and 2; and calculated the demands faced by firms 1 and 2 as $S_1(t) = M(t)\bar{l}(t)$ and $S_2(t) = M(t) - S_1(t)$, respectively. The two firms can increase the total demand for their products over time by investing on advertising that evolves according to the differential equation that $\dot{M}(t) = \alpha[u_1(t) + u_2(t)] - \delta M(t)$, where $u_i(t)$ is firm i's advertising effort, $\alpha > 0$ measures the advertising effectiveness, and δ is the depreciation rate of the demand.

Similar to Bertuzzi and Lambertini [9], Cellini and Lambertini [18] considered a Cournot competition setting where $n \geq 2$ firms sell a homogenous product at time t. All firms were assumed to incur an identical marginal production cost which was normalized to zero. The *inverse* demand function $S(t)$ satisfies the equation that $p(t) = [p_R(t) - S(t)]^{1/\alpha}$, where $p(t)$ is the price of the product, $p_R(t)$ is consumers' reservation price, $S(t) = \sum_{i=1}^{n} S_i(t)$ represents the aggregate quantity of the product, and $\alpha \in (0, +\infty)$ determines the curvature of the demand. The function $S(t)$ is convex when $\alpha > 0$, is linear when $\alpha = 1$, and is concave when $\alpha \in (1, +\infty)$.

The n firms can increase the value of $p_R(t)$ by investing on advertising that follows the following evolution dynamics: $\dot{p}_R(t) = \sum_{i=1}^{n} u_i(t) - \delta p_R(t)$ with $p_R(0) = \bar{p}_R$. Each firm maximizes its individual discounted profit. Solving the differential game, Cellini and Lambertini derived the open-loop and the closed-loop equilibria, and showed that the properties of the equilibria are dependent on the curvature of the demand function. They also found that the firms' advertising efforts in the open-loop equilibrium are greater than those in the closed-loop equilibrium.

2.2.3 The Advertising Strategies with a Competitor's Negative Interference

Viscolani and Zaccour [160] examined a duopoly problem where each firm's current sales are proportional to its goodwill stock, which evolves positively related to the firm's own advertising effort and negatively related to that of its competitor. The goodwill dynamics for firm i ($i = 1, 2$) was given as $\dot{A}_i(t) = \gamma_i u_i(t) - \eta_{ij}\gamma_j u_j(t) - \delta A_i(t)$ with $A_i(0) = \bar{A}_i > 0$, for $j = 1, 2$ and $j \neq i$, where $\gamma_i \geq 0$ is the advertising effort efficiency, η_{ij} is the interference factor of firm j's advertising on firm i's goodwill evolution, and $\delta > 0$ is a decay parameter. The demand rate for each brand at time t was assumed to be proportional to its goodwill stock; that is, $S_i(t) = \beta A_i(t)$. Each firm makes an optimal decision to maximize its discounted profit. Viscolani and Zaccour showed that, when both firms are strong, a unique Nash equilibrium is the same as that in the absence of interference from the competitor's advertising.

2.3 The Extended Nerlove-Arrow Models for Dynamic Advertising Problems in Supply Chain Analysis

In the past two decades, a number of researchers have used the classic N-A model and its generalizations to investigate the dynamic advertising problems arising in supply chain operations. Since the interaction between/among the firms in supply chains is different from that in the duopoly and oligopoly settings, the advertising policies may vary in different systems.

2.3.1 Supply Chain Analysis with Only Dynamic Advertising Decision

As a seminal publication pertaining to supply chain analysis with the dynamic advertising decision, Chintagunta and Jain [21] used the classic N-A model in (1) to examine dynamic advertising strategies in a two-echelon supply chain, where a manufacturer makes a product and sells it through a retailer. The stocks of goodwill for the manufacturer and the retailer—which were denoted by $A_M(t)$ and $A_R(t)$, respectively—were described by the following classic N-A model: $\dot{A}_i(t) = u_i(t) - \delta A_i(t)$ with $A_i(0) = \bar{A}_i$, for $i = M, R$; and the sales were given as $S(A_M(t), A_R(t))$. Assuming a specific quadratic sales response function and a linear cost function, Chintagunta and Jain derived the equilibrium effort levels for the cases with and without coordination between the two firms.

Different from Chintagunta and Jain [21], Jørgensen, Sigué, and Zaccour [71] examined a supply chain involving a manufacturer and a retailer who "simultaneously" (with no communication) invest on the short-term and long-term advertising. The manufacturer's (retailer's) short-term and long-term advertising effort rates are denoted by $u_S(t)$ and $u_L(t)$ ($\upsilon_S(t)$ and $\upsilon_L(t)$), respectively. Moreover, the manufacturer decides to absorb $\phi_S(t) \in [0, 1]$ of the retailer's short-term advertising cost and also share $\phi_L(t) \in [0, 1]$ of the retailer's long-term advertising cost. The authors assumed that the goodwill stock is dependent on the long-term advertising effort rather than the short-term advertising effort; that is, the differential function of the goodwill stock was developed as $\dot{A}(t) = \lambda_M u_L(t) + \lambda_R \upsilon_L(t) - \delta A(t)$ with $A(0) = A_0 \geq 0$, where λ_M, λ_R, and δ are nonnegative constants. The time-dependent demand function was assumed as $S(u_S(t), \upsilon_S(t), A(t)) = [\alpha_M u_S(t) + \alpha_R \upsilon_S(t)]\sqrt{A(t)}$, where $\alpha_M \geq 0$ and $\alpha_R \geq 0$ represent the effectiveness of the manufacturer's and the retailer's short-term advertising, respectively. Jørgensen, Sigué, and Zaccour considered a leader-follower game with the manufacturer as the leader and the retailer as the follower, and derived Stackelberg equilibrium strategies for the two firms.

Similar to Jørgensen, Sigué, and Zaccour [71], Jørgensen, Taboubi, and Zaccour [73] considered a cooperative advertising problem in a supply chain where a manufacturer invests on advertising to build a stock of goodwill and the retailer invests on promoting the manufacturer's product. Both the goodwill and the promotion activity were assumed to impact the sales, as described by the function that $S(u(t), A(t)) = \beta u(t) + \lambda A(t) - \gamma [A(t)]^2$, where $\beta > 0$ represents the effect of the retailer's promotion on sales, $u(t)$ is the retailer's promotion effort, $\lambda > 0$ and $\gamma > 0$ capture the effect of goodwill on sales, and $A(t)$ is the manufacturer-created goodwill stock, which evolves according to the classic N-A model in (1). The authors investigated (i) a "simultaneous-move" game where the manufacturer and the retailer make their decisions with no communication and (ii) a leader-follower game where the manufacturer first makes its decision as the leader and the retailer then responds as the follower. For both games, the authors derived the stationary Markovian strategies, which show that, no matter whether or not the goodwill has a decreasing marginal effect on sales, the cooperative advertising program can always coordinate the supply chain.

Differing from Jørgensen, Sigué, and Zaccour [71] and Jørgensen, Taboubi, and Zaccour [73] where the retailer's promotion effort does not impact the goodwill stock, Jørgensen, Taboubi, and Zaccour [74] assumed that the retailer's promotion effort for the sake of increasing the sales revenue can damage the manufacturer's brand image. The goodwill stock evolves according to the following dynamics: $\dot{A}(t) = \alpha u_M(t) - \beta u_R(t) - \delta A(t)$ with $A(0) = A_0 \geq 0$, where α and β are positive constants denoting the impacts of the manufacturer's advertising effort $u_M(t)$ and the retailer's promotion effort $u_R(t)$, respectively, and δ means the decay rate of the brand image. The rate of sales revenue was given as $S(u_R(t), A(t)) = \gamma u_R(t) + \theta A(t)$, where $\gamma > 0$ and $\theta > 0$ measure the effects of promotion and brand image on the sales revenue, respectively. The authors derived Nash equilibria for both the case where the retailer is myopic and the case where the retailer is far-sighted, and also obtained Stackelberg equilibrium for a leader-follower game model where the manufacturer acts as the leader and cooperates with the retailer (who acts as the follower) by sharing a part of the retailer's promotion cost.

Generalizing the models by Jørgensen, Sigué, and Zaccour [71] and Jørgensen, Taboubi, and Zaccour [74], Jørgensen and Zaccour [77] considered a two-echelon supply chain consisting of a manufacturer and n retailers whose promotion efforts negatively affect the manufacturer's brand image. Assuming that, at time t, retailer $i \in \{1, 2, \cdots, n\}$ controls its local promotion rate $\upsilon_i(t)$, and the manufacturer invests $u_M(t)$ on the advertising, Jørgensen and Zaccour developed the differential functions for the evolution of the brand image $A(t)$ and the stock of "cumulative overall promotion" $K(t)$ as $\dot{A}(t) = u_M(t) - \beta K(t)$ with $A(0) = A_0 > 0$ and $\dot{K}(t) = nu_R(t) - \mu K(t)$ with $K(0) = K_0 > 0$, respectively. Note that, in the differential functions, $\mu > 0$ is the decay rate of cumulative overall promotion, and $nu_R(t) = \sum_{i=1}^{n} \upsilon_i(t)$ when $\upsilon_i(t) = u_R(t)$ under the assumption that n retailers are symmetric. The demand rate for the brand was assumed as $S(u_R(t), A(t)) = \gamma u_R(t) + \theta A(t)$, where $\gamma > 0$ and $\theta > 0$ denote the effectiveness of the promotion and that of brand image on the demand rate, respectively. Jørgensen and Zaccour then derived Nash equilibrium for the "simultaneous-move" game, and also obtained Nash equilibrium for the game where the manufacturer designs and announces an incentive scheme to the retailers.

2.3.2 Supply Chain Analysis with Dynamic Advertising and Pricing Decisions

In practice, supply chain members may control both the advertising investment and the selling price to improve the sales and their profits. As an extension of the model by Chintagunta and Jain [21], Jørgensen and Zaccour [75] considered a dynamic advertising model to investigate a supply chain in which, at time t, the manufacturer and the retailer determine their advertising efforts $u_M(t)$ and $u_R(t)$, respectively, and the retailer also decides on the retail price $p(t)$. Similar to [71], Jørgensen and Zaccour [75] characterized the goodwill stock as $\dot{A}(t) = u_M(t) + u_R(t) - \delta A(t)$ with $A(0) = A_0$. In addition, the sales were calculated by using the equation that $S(t) = [\alpha - \beta p(t)]\{g_1 A(t) - g_2 [A(t)]^2 /2\}$ where α, β, g_1, and g_2 are all positive constants.

Different from [75], Jørgensen, Sigué, and Zaccour [72] considered a supply chain involving a manufacturer and a retailer, where the goodwill stock depends on advertising effort of the manufacturer rather than that of the retailer, and examined which supply chain member should lead the channel. Similar to the classic N-A model in (1), the dynamics of the goodwill stock was given as $\dot{A}(t) = u_M(t) - \delta A(t)$ with $A(0) = A_0$. The sales response function was specified as $S(t) = u_R(t)[\alpha - \beta p(t)]\sqrt{A(t)}$, where α and β are two positive constants. The authors analyzed Nash and Stackelberg equilibria, and found that the supply chain inefficiency is reduced and consumers thus benefit more, as the manufacturer takes the leadership.

Jørgensen and Zaccour [76] analyzed supply chain coordination over time, assuming that both the manufacturer's and the retailer's advertising efforts impact the goodwill stock. Accordingly, as given in [71], the differential equation for the goodwill stock was written as $\dot{A}(t) = \lambda_M u_M(t) + \lambda_R u_R(t) - \delta A(t)$ with $A(0) = A_0 \geq 0$, where λ_M and λ_R are the manufacturer's and the retailer's advertising efficiency parameters, respectively. Using the demand rate given by Jørgensen and Zaccour [75], Jørgensen and Zaccour [76] found that the Pareto-optimal solution can be realized if each firm employs a marketing strategy that is a linear function of the other firm's decision variable and both firms jointly implement their strategies.

Assuming that both the manufacturer's and the retailer's advertising efforts affect the goodwill stock in a positive manner, Zaccour [164] developed a differential game model of the goodwill stock, which is similar to that in [75], and developed a linear demand function as $S(t) = \mu + A(t) - \alpha p(t)$, where μ, $\alpha > 0$. Zaccour derived the conditions under which the manufacturer can utilize a two-part tariff to achieve the vertically integrated channel solution in a static model, and showed that the results in a dynamic advertising model are different from those in the static model.

2.3.3 Supply Chain Analysis with Dynamic Advertising Models for Competing National and Store Brands

Several recent publications are concerned with the optimal advertising decisions when a national brand competes with a retailer's store brand. For example, Amrouche, Martín-Herrán, and Zaccour ([1] and [2]) analyzed such a competition problem when the advertising effort for a brand negatively affects the goodwill stock for the other brand. Letting NB and SB denote the national brand and the store brand, respectively, the authors characterized the goodwill stock for brand $i \in \{NB, SB\}$ by using the dynamics that $\dot{A}_i(t) = u_i(t) - \delta A_i(t) - k_i u_j(t)$ with $A_i(0) = \bar{A}_i \geq 0$, for $i, j \in \{NB, SB\}$ and $j \neq i$, where $\delta > 0$ denotes the decay rate and the parameter k_i represents the vulnerability of brand i resulting from the advertising effort for brand j.

Amrouche, Martín-Herrán, and Zaccour [1] gave the demand function for brand i as $S_i(t) = \beta_i + A_i(t) - p_i(t) + \psi p_j(t)$, where $\beta_i + A_i(t)$ reflects the market potential and $\psi \in [0, 1)$ represents the degree of substitutability between the two brands. The feedback Stackelberg pricing and advertising strategies were derived for the case that the NB manufacturer acts as the leader. Amrouche, Martín-Herrán, and Zaccour [2] developed the demand function for brand i as $S_i(t) = \alpha_i A_i(t) - p_i(t) + \psi_i p_j(t)$, where $\psi_i \in (0, 1)$ is a parameter capturing the cross-price effect on demand. They obtained the feedback Nash equilibrium, and examined the effect of each brand's goodwill on the equilibrium pricing and advertising strategies.

Similar to the above two papers, Karray and Martín-Herrán [81] examined the carryover effects of brand advertising over time for both the national brand and the store brand, taking into account both the complementary and the competitive roles of advertising. Karray and Martín-Herrán used a goodwill dynamics similar to that in Chintagunta [20], i.e., $\dot{A}_i(t) = \delta\sqrt{u_i(t)} - \lambda A_i(t)$ with $A_i(0) = \bar{A}_i > 0$, for $i \in \{NB, SB\}$, where δ and λ are two positive constants; and, the two authors constructed the demand function as $S_i(t) = b_i + \psi_i A_i(t) + \theta_i A_j(t) - p_i(t) + \alpha[p_j(t) - p_i(t)]$, for $i, j \in \{NB, SB\}$ and $i \neq j$, where $p_i(t)$ is the retail price for brand i; b_i is a positive constant that represents the baseline sales for brand i; ψ_i is a positive constant representing the effect of the goodwill for brand i on demand for the brand; $\alpha \in (0, 1)$ denotes the cross-price competitive effect; and θ_i can be positive or negative, which is in accordance with the complementary or competitive effect of a brand's advertising on the demand for the other brand. Karray and Martín-Herrán showed that the relationship between advertising and pricing in the supply chain is dependent on the nature of the advertising effects.

2.3.4 Supply Chain Analysis with Dynamic Models Involving Advertising and Other Attributes

We now review recent publications that contain dynamic advertising models with other attributes (i.e., quality, strategic interactions, and space) in supply chain analysis.

Supply Chain Analysis with Advertising and Quality Improvement Giovanni [59] assumed that the build-up of goodwill is made through both advertising and quality improvement. The author investigated two settings for a supply chain consisting of a manufacturer and a retailer. Specifically, in the first (non-cooperative) setting, only the retailer makes its advertising effort and the manufacturer determines its quality improvement and wholesale price; in the second (cooperative) setting, the manufacturer supports the retailer's advertising but reduces its investment in quality.

Suppose that the retail price is a function of the wholesale price, i.e., $p(w(t)) = \eta w(t)$, where $\eta > 1$ is the retailer's constant mark-up applied to the wholesale price. Letting $q(t)$ denote the manufacturer's quality improvement rate, Giovanni developed the goodwill dynamics and the sales function as,

$$\dot{A}(t) = \varepsilon u(t) + \gamma q(t) - \delta A(t) \text{ with } A(0) = A_0 > 0 \quad \text{and} \quad S(t) = \alpha - \beta\eta w(t) + \theta A(t),$$

where $\varepsilon > 0$ and $\gamma > 0$ represent the marginal effects of advertising and quality on the goodwill, $\delta > 0$ is a parameter reflecting the forgetting effect of goodwill, $\alpha > 0$ is the market potential, and $\beta > 0$ and $\theta > 0$ denote the effects of pricing and goodwill on the current sales, respectively.

The manufacturer's production cost increases with the quality improvement, i.e., $C(q(t)) = cq(t)$. Assume that the manufacturer is the decision leader and the retailer is the follower in a Stackelberg game. Using the quadratic forms for the advertising cost and quality improvement cost, Giovanni obtained the Stackelberg equilibria for both the non-cooperative and the cooperative settings, and found that the manufacturer has an incentive to cooperate only when advertising significantly improves the goodwill, whereas the retailer is always better off in the cooperative setting.

Supply Chain Analysis with Strategic Interactions in Traditional Franchise Systems Sigué and Chintagunta [144] considered a two-stage advertising game involving a franchisor and two franchisees, and examined who should undertake the promotion and brand-image advertising. Accordingly, the authors investigated three cases. In Case I, the franchisor delegates both the promotional and the brand-image advertising decisions to two franchisees, sharing a portion of their advertising expenses through a cooperative advertising program where each franchisee determines the investment levels in the promotional and the brand-image advertising. Letting $u_i(t)$ and $v_i(t)$ $(i = 1, 2)$ denote the promotional and the brand-image advertising efforts, Sigué and Chintagunta described the goodwill dynamics and total sales as $\dot{A}(t) = v_1(t) + v_2(t) - \delta A(t)$ and $S_i(t) = \{\alpha u_i(t) + \beta[u_i(t) - u_j(t)]\}\sqrt{A(t)}$, for $i, j = 1, 2$ and $i \neq j$, with $A(0) = A_0 \geq 0$.

In Case II, the franchisor controls the brand-image advertising decision and bears a part of two franchisees' promotional advertising investments, and each franchisee decides on its effort for the promotional advertising. Similar to the classic N-A model in (1), the goodwill dynamics for the second case was developed as $\dot{A}(t) = v_B(t) - \delta A(t)$, where $v_B(t)$ is the franchisor's investment on the brand-image advertising. Note that the sales dynamics for Case II is the same as that in Case I. In Case III, the franchisor determines all the advertising decisions and the franchisees play no role in the advertising program. The goodwill dynamics and sales are both the same as those in Case II.

Martín-Herrán, Sigué, and Zaccour [98] investigated how price competition and advertising spillover affect franchisees' decisions to cooperate and the franchisor's choice of contracts. They considered a franchising system consisting of one franchisor and two symmetric franchisees, where the franchisor decides the wholesale price $w(t)$ and the advertising effort $u_M(t)$, and franchisee i determines the retail price $p_i(t)$, $i = 1, 2$. The goodwill dynamics is the same as the classic N-A model in (1), i.e., $\dot{A}(t) = ku_M(t) - \delta A(t)$ with $A(0) = A_0 > 0$. Denoting by $u_i(t)$ franchisee i's investment on local advertising activity, the authors characterized franchisee i's demand in a linear form as $S_i(t) = \alpha + \theta A(t) - p_i(t) + \gamma p_j(t) + \lambda u_i(t) + \mu u_j(t)$, for $i, j = 1, 2$ and $i \neq j$, where $\alpha, \theta > 0$, $0 \leq \gamma < 1$, $\lambda \geq 0$; and the term $\alpha + \theta A(t)$ means the market potential. The effect of franchisee i's local advertising $u_i(t)$ on demand was assumed to be lower than the effect of its price $p_i(t)$ but greater than or equal to the effect of franchisee j's advertising effort $u_j(t)$, i.e., $|\mu| \leq \lambda < 1$. When $\mu > 0$, there exists advertising spillover. The authors derived the equilibria for both the case when the two franchisees cooperate and the case when they do not cooperate, and found that the franchisees' cooperative behavior depends on the franchise contracts.

Supply Chain Analysis with Dynamic Advertising and Shelf-Space Allocation Martín-Herrán, Taboubi, and Zaccour [99] examined a retailer's strategy of allocating its shelf space between two manufacturers who make competing brands. They accordingly considered a supply chain involving two competing manufacturers and one retailer, where each manufacturer intends to gain more allocation by investing on advertising to improve its brand's goodwill and thus stimulate the demand for its product. The goodwill stock for manufacturer i ($i = 1, 2$) was described, similar to the classic N-A model in (1), as $\dot{A}_i(t) = \alpha_i u_i(t) - \delta A_i(t)$ with $A_i(0) = \bar{A}_i > 0$, where α_i is a positive parameter that measures the advertising efficiency of manufacturer i, allowing the effects between two brands to be asymmetric; the parameter δ is the decay rate and is identical for both brands.

The retailer has limited space for the allocation between the two brands. Assume that the space allocated to brand i ($i = 1, 2$) is $\beta_i(t)$ such that $\beta_1(t) + \beta_2(t) = 1$. The demand function $S_i(t)$ for brand i was developed as $S_i(t) = \beta_i(t)[a_i A_i(t) - b_i \beta_i(t)/2]$, where $a_i, b_i > 0$. The authors derived the time-consistent open-loop Stackelberg equilibrium for a leader-follower game where the manufacturers are the leaders and the retailer is the follower.

2.4 General Remarks

In this section, we review major N-A extensions that were proposed after 1995. These N-A models can be generally divided into two categories: (i) the N-A models in the monopoly, duopoly, and oligopoly settings; for a summary, see Table B in the online appendix where we find that most models are concerned with the dynamic advertising policy for a single firm (in a monopoly setting); and (ii) the N-A models in the supply chain setting; for a summary, see Table C in the online appendix where we learn that most models focus on the supply chain involving a manufacturer and a retailer. It thus follows from our survey that, in the future, we should pay more attention to the dynamic advertising problems with horizontal competition between two, or among three or more firms in the duopoly, oligopoly, or supply chain setting.

3 Vidale-Wolfe Model and Extensions

The dynamic models reviewed in this section concern the impact of advertising on sales that generally persists beyond the current period with a diminishing effect. The direct relation between the rate of change in sales and the advertising carryover effect was often characterized by a differential or difference equation, as discussed by Sethi [138]. The earliest model incorporating such an advertising effect is called the V-W model, which was constructed by Vidale and Wolfe in [159]. Letting λ be the exponential sales decay constant, M the saturation level, and r the response constant, Vidale and

Wolfe [159] characterized the sales rate $S(t)$ at time t as,

$$\dot{S}(t) = ru(t)[M - S(t)]/M - \lambda S(t), \tag{6}$$

where $u(t)$ denotes the rate of advertising expenditure as in Section 2, $[M - S(t)]/M$ is the fraction of potential customers, and $\lambda S(t)$ is the number of lost customers. As the above differential sales equation indicates, the change in the rate of sales, $\dot{S}(t)$, is positively related to the advertising effort $u(t)$.

The above basic V-W model has been widely extended to analyze various business- and economics-related problems. Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138] reviewed the V-W extensions that were published before 1995. For a summary of major before-1995 models, see Table D in the online appendix. Next, we review the recent extended V-W models that were published after 1995 to investigate dynamic advertising problems.

3.1 The Extended Vidale-Wolfe Models for Dynamic Advertising Problems in the Monopoly Setting

We review recent V-W extensions that were published after 1995, which are classified into the following four categories: the pulsing advertising models with S-shaped response functions, the integrated marketing communications with uncertainty, the optimal advertising and pricing decisions in new product adoption, and the infinite-horizon optimal advertising for durable goods.

3.1.1 The Advertising Pulsing Models with S-shaped or Concave Response Functions

Among the before-1995 V-W extensions reviewed by Feichtinger, Hartl, and Sethi [44], Mahajan and Muller [110] used an S-shaped advertising effectiveness function to examine whether an advertising pulsing policy is superior to an even policy. Similar to Mahajan and Muller [110], Mesak [100] generalized the V-W model in (6) and investigated the advertising pulsing policy using both S-shaped and concave advertising response functions as follows:

$$\dot{S}(t) = \{\lambda + r[u(t)]^{\delta}/M\}[\Lambda(u(t)) - S(t)] = r[u(t)]^{\delta}[M - S(t)]/M - \lambda S(t), \tag{7}$$

where δ is a positive constant and $\Lambda(u(t)) \equiv r[u(t)]^{\delta}/\{\lambda + r[u(t)]^{\delta}/M\}$ represents the steady state advertising response function. Note that $\Lambda(u(t))$ is concave for $0 < \delta \leq 1$ and is S-shaped for $\delta > 1$. Incorporating the advertising wearout effect into (7), Mesak found the optimal sales response $S(t)$ to the even policy and the pulsing policy. For both the S-shaped and concave response functions, Mesak derived the conditions under which the pulsing policy is superior to the even policy in the presence of advertising wearout.

Replacing $r[u(t)]^{\delta}/M$ in (7) with the general function $f(u(t))$, Mesak [103] developed a non-discounted measure of performance to investigate the impact of initial sales rate on the performance of a variety of discrete and piecewise-continuous advertising policies for a finite planning horizon. Mesak showed that, irrespective of the shape of the advertising response function, a firm's advertising effort for a Blitz policy is much different from that for an advertising pulsing/maintenance policy, when the initial sales rate does not equal zero at the beginning of the planning period. Mesak also found that the pattern of the optimal advertising policy in the presence of an initial sales rate significantly differs from that when an initial sales rate is absent.

Generalizing Mesak's model in [103], Mesak and Ellis [108] investigated the advertising pulsing policy when the advertising market potential function $M(u(t))$ is concave and the general advertising response function $f(u(t))$ is concave or linear, and showed that a pulsing policy is superior to an even policy.

Different from the above models, Feinberg [47] assumed that the market share evolves according to the equation that $\dot{S}(t) = f(u(t))a(u(t)) - b(u(t))$, where $f(u(t))$ is an S-shaped function. The S-shapeness of the function $f(u(t))$ can be decoupled from the properties of the "acceleration" function $a(u(t))$ and the "decay" function $b(u(t))$ of sales. Feinberg provided a rational as to when the non-constant advertising policy is optimal, using the classic phase-plane analysis with unspecified nonlinearity in the response function.

3.1.2 The Integrated Marketing Communications with Uncertainty

Prasad and Sethi [128] considered firms' marketing communications decisions in markets with uncertainty and competition, where firms often utilize multiple communications instruments such as TV, print, concurrent advertising, promotion, etc., in their marketing campaign. Prasad and Sethi extended Sethi's model [139] (which is presented in Table D in the online appendix) to incorporate multiple advertising decisions with the evolution of the market share in the following stochastic manner: $dx(t) = [U(u(x(t)), v(x(t)))\sqrt{1 - x(t)} - kx(t)]dt + \sigma(x(t))d\xi(t)$, where k, $\sigma(x(t))$, and $d\xi(t)$ are defined as in Table D; $u(t)$ and $v(t)$ are the advertising rates at time t for two types of advertising instruments; and, $U(u(x(t)), v(x(t))) \equiv \rho_u u(x(t)) + \rho_v v(x(t)) + \hat{k}\sqrt{u(x(t))v(x(t))}$, with the advertising effectiveness parameters $\rho_u > 0$ and $\rho_v > 0$, and the synergy parameter $\hat{k} \geq 0$.

3.1.3 The Optimal Advertising and Pricing Decisions in the New-Product Adoption

Sethi et al. [140] extended Sethi's model in [139] (given in Table D) to consider optimal advertising and pricing decisions for a firm who introduces a new product. Let $S(t)$ denote the cumulative sales at time t, $p(t)$ denote the price charged at time t, and $D(p(t))$ denote the price-dependent demand with $D'(p) < 0$. Sethi et al. [140] developed an advertising dynamics model as $\dot{S}(t) = ru(t)D(p(t))\sqrt{1 - S(t)} - \lambda S(t)$, where r and λ are defined as in the V-M model in (6). Similar to Sethi [139], the advertising cost was assumed as $[u(t)]^2$ and the demand $D(p(t))$ was supposed to be either in the linear function form that $D(p(t)) = 1 - \eta p(t)$ with $\eta > 0$ or in an isoelastic function form $D(p(t)) = [p(t)]^{-\eta}$ with $\eta > 1$.

3.2 The Extended Vidale-Wolfe Models for Dynamic Advertising Problems with Horizontal Competition

Similar to Section 2.2, we review the V-W extensions published after 1995, which concern dynamic advertising models in the presence of horizontal competition.

3.2.1 The Advertising Competition under Uncertainty

Prasad and Sethi [127] analyzed firms' optimal advertising decisions in a duopoly market, where each firm's market share is dependent on its own and its competitor's advertising decisions. Extending Sethi's model in [139] (given in Table D) to a duopoly setting (where there are firms 1 and 2), Prasad and Sethi proposed the dynamics model for firm i ($i = 1, 2$) as,

$$dx_i(t) = \{\rho_i u_i(x_i(t))\sqrt{1 - x_i(t)} - \rho_j u_j(x_i(t))\sqrt{x_i(t)} - k[2x_i(t) - 1]\}dt + \sigma(x_i(t))d\xi(t), \quad (8)$$

where $x_i(t)$ represents firm i's market share such that $x_1(t) + x_2(t) = 1$, for $j = 1, 2$ and $j \neq i$. Note that, if $\sigma(x_i(t)) = 0$, then the dynamics model in (8) is similar to Sorger's model [146], which is regarded as an extension of the Lanchester model in Section 4. Prasad and Sethi showed that $x_1(t) = 0$ and $x_1(t) = 1$ are natural boundaries for the solutions of the dynamics in (8). Prasad and Sethi obtained the closed-loop Nash equilibrium for the differential game between firms 1 and 2.

3.2.2 The Advertising Competition in a Dynamic Oligopoly Setting

Erickson [41] generalized Sethi's model in [139] to an oligopoly setting where the advertising dynamics was developed as $\dot{S}_i(t) = \beta_i a_i \sqrt{M - \sum_{j=1}^{n} S_j(t)} - \rho_i S_i(t)$, for $i = 1, 2, \ldots, n$. This model can be regarded as a V-W extension involving competitors' advertising efforts on attracting the untapped potential, allowing for an expansion of the total sales of the oligopolistic competitors. Erickson derived a feedback Nash equilibrium for both the symmetric and asymmetric cases.

Extending the model in [41] where each firm considers its advertising policy for a single brand, Erickson [40] investigated the firms' optimal advertising decisions in an oligopoly market, where each firm may sell multiple brands. Suppose that the market involves n competing firms and firm i $(i = 1, 2, ..., n)$ owns J_i brands—namely, b_j $(j = 1, \ldots, J_i)$. Erickson extended the V-W model in (6) to describe the advertising dynamics for the sales of brand b_j of firm i as $\dot{S}_{ij}(t) = r_{ij} u_{ij}(t) \sqrt{M - \sum_{k=1}^{n} \sum_{l=1}^{J_k} S_{kl}(t)} - \lambda_{ij} S_{ij}(t)$, where $S_{ij}(t)$ and $u_{ij}(t)$ represent the sales and advertising effort of firm i for brand b_j, and r_{ij} and λ_{ij} are defined similarly as the parameters r and λ in the V-W model. Using a discounted profit similar to that given by Vidale and Wolfe [159], Erickson provided the closed-loop Nash equilibrium for a differential game.

3.2.3 The Dynamic Advertising Model with Joint Advertising and Pricing Decisions

Krishnamoorthy et al. [86] extended the dynamic model proposed by Sethi et al. [140] to a duopoly setting, where two firms compete by making both the advertising and pricing decisions. The sales dynamics equation was given as $\dot{S}_i(t) = r_i u_i(t) D_i(p_i(t)) \sqrt{M - S_i(t) - S_j(t)}$, for $i, j = 1, 2$ and $i \neq j$, where $D_i(p_i(t))$ is the price-dependent demand faced by firm i. Using the linear and isoelastic demand function forms as in Sethi et al. [140], Krishnamoorthy et al. [86] obtained the closed-form Nash equilibria for both the game with symmetric competitors and that with asymmetric competitors.

3.3 The Extended Vidale-Wolfe Models in Supply Chain Analysis

Using Sethi's model [139] (given in Table D in the online appendix), He et al. [66] considered cooperative advertising and pricing decisions for a supply chain, where a manufacturer first decides on the wholesale price $w(t)$ and the participation rate $\theta(t)$ for a cooperative advertising support scheme and a retailer then determines the retail price $p(t)$ and its advertising effort $u(t)$ for the supply chain. Assuming that the retailer's advertising cost is $[u(t)]^2$, the authors calculated the manufacturer's and the retailer's expenditures at time t as $\theta\,[u(t)]^2$ and $(1-\theta)\,[u(t)]^2$, respectively. He et al. attained the optimal advertising and pricing decisions for the manufacturer and the retailer, and derived the condition under which it is optimal for the manufacturer to offer the cooperative advertising scheme.

He et al. [65] investigated a cooperative advertising scheme for a supply chain involving a manufacturer selling its product through retailer 1, who competes with retailer 2—an independent retailer selling a substitute product. The manufacturer awards a subsidy to retailer 1 when the retailer's advertising level exceeds a certain threshold. He et al. analyzed a leader-follower game where the manufacturer first sets the subsidy rate (per dollar unit of the advertising cost) $\theta(t)$ to retailer 1, and retailers 1 and 2 then determine their advertising levels $u_1(t)$ and $u_2(t)$, respectively. The advertising dynamics for retailer 1 was developed as $\dot{x}(t) = \rho u_1(t)\sqrt{1-x(t)} - \rho u_2(t)\sqrt{x(t)} - kx(t) + k[1-x(t)]$ with $x(0) = x_0 \in [0,1]$, where $x(t)$ and $1-x(t)$ represent retailer 1's and retailer 2's market shares, respectively. The Stackelberg equilibrium was attained for the differential game.

Chutani and Sethi [23] extended the above problem in He et al. [65] to address a cooperative advertising problem for a supply chain where the manufacturer shares the advertising cost with one or both of the competing retailers (retailers 1 and 2). Chutani and Sethi considered a Stackelberg differential game where the manufacturer determines its subsidy rates for the two retailers, and the retailers then respond by choosing their advertising efforts in a Nash differential game setting. The dynamics of retailer i's ($i = 1,2$) market share was modeled as, $\dot{x}_i(t) = \rho_i u_i(t)\sqrt{1 - x_i(t) - x_j(t)} - k_i x_i(t)$, for $j = 1,2$ and $j \neq i$.

Differing from the above models, Ouardighi et al. [120] developed a differential game model to investigate a coordination problem involving operations and marketing activities that are performed by a manufacturer and a retailer in a supply chain. The manufacturer determines the production volume, production process improvement, and national advertising efforts, while the retailer decides on its purchase volume and the retail price. The dynamic equations governing the evolution of the product quality $Q(t) \in (0,1]$ and consumer sales $S(t)$ are given as,

$$\dot{Q}(t) = \vartheta(t)[1 - Q(t)] \text{ and } \dot{S}(t) = \rho u(t)[(\alpha - \beta p(t))^{\gamma} - S(t)] - \delta[1 - Q(t)]S(t),$$

where ρ, α, and β are positive constants; $\gamma \in (0,1)$ and $\delta \in (0,1)$ are two constants; $\vartheta(t) \geq 0$ is the manufacturer's rate of quality improvement effort; $p(t) \in [0, \alpha/\beta)$ is the retail price. Ouardighi et al. [120] derived the optimal decisions for both the manufacturer and the retailer, and also showed that a revenue sharing contract can coordinate the decentralized supply chain.

3.4 General Remarks

We learn from the above review that, after 1995, the V-W modeling approach was first adopted to construct sales-advertising response models, and was then extended by, e.g., including the pricing decision. For a brief summary, see Table E in the online appendix where we find that, except for the models developed by Sethi et al. [140] and Ouardighi et al. [120], no publications incorporated non-advertising market instruments (i.e., pricing, quality, etc.) into the V-W models, which may be a research direction in the future. Moreover, our summary in Table E indicates that, similar to our summary of the N-A models in Section 2.4, a few recent publications have used the V-W model and its extensions to investigate the dynamic advertising problems in supply chain systems, which should also be a future research direction.

4 Lanchester Model and Extensions

Kimball [82] introduced the Lanchester model to analyze competitive advertising problems, which resemble Lanchester's models of warfare. For a duopoly setting where two firms compete for the market share with their advertising efforts, we can characterize firm i's ($i = 1,2$) market share at time t, $x_i(t)$, in the following Lanchester function form:

$$\dot{x}_i(t) = \beta_i u_i(t)x_j(t) - \beta_j u_j(t)x_i(t) = \beta_i u_i(t)[1 - x_i(t)] - \beta_j u_j(t)x_i(t), \quad \text{for } j = 1,2 \text{ and } j \neq i, \quad (9)$$

where $x_j(t) = 1 - x_i(t)$ is firm j's market share; β_i and β_j are two positive parameters measuring firms i's and j's advertising effectiveness, respectively; $u_i(t)$ and $u_j(t)$ are firms i's and j's advertising rates.

When we compare the basic Lanchester model in (9) with the basic V-W model in (6), we find that if $\beta_j u_j$ in (9) is assumed to be a constant and the subscript is removed, then the basic Lanchester model can be reduced to Sethi's model in [136]—which is an extension of the V-W model, as indicated by Table D (in the online appendix). Therefore, the Lanchester model could be regarded as an extension of the V-W model to the duopoly setting with advertising competition. In the past several decades, a number of Lanchester extensions have been developed to examine dynamic advertising problems involving multi-firm competition. Erickson [38] and Feichtinger, Hartl, and Sethi [44] reviewed the relevant models proposed before 1995; for our brief summary, see Table F in the online appendix. In this section, we review the extended Lanchester models that were published after 1995.

4.1 The Extended Lanchester Models for the Dynamic Advertising Competition Problems in the Duopoly Setting

We now review the extended Lanchester models that were published after 1995 to investigate the dynamic advertising problems in the duopoly setting.

4.1.1 Advertising Competition in a Market with a Fixed or Dynamic Size

Using the Lanchester model in (9) to capture the dynamics of the market share when two firms compete in a market, Fruchter and Kalish [55] derived and explained the open- and the closed-loop strategies for the firms. The authors also showed that the closed-loop solution may be a global Nash equilibrium.

Extending the Lanchester model in (9) and Deal's model [26], Wang and Wu [162] proposed a new differential game model to investigate an advertising competition problem with non-durable products. Their extended Lanchester model was developed as $\dot{x}_i(t) = \beta_i u_i(t)[1 - x_i(t)] - \beta_j u_j(t) x_i(t) - \gamma_i x_i(t)$ with $x_i(0) = \bar{x}_i$, for $i, j = 1, 2$ and $i \neq j$, where $x_i(t) = S_i(t)/M$, and M is the market size. Suppose that firm i maximizes the sum of its cumulative profit over the planning interval $[0, T]$ and a valuation of the ending market share; that is, firm i's maximization problem was given as, $\max_{u_i(t)\geq 0} J_i = \int_0^T (M p_i x_i(t) - [u_i(t)]^2) dt + g_i x_i(T)$, where p_i is firm i's unit profit margin and g_i is firm i's valuation of market share at time T. For the differential game, Wang and Wu derived the necessary and sufficient conditions for the open- and the closed-loop Nash equilibria.

Extending the Lanchester model in (9), Jarrar et al. [69] developed a new differential game model with square advertising effect as, $\dot{x}(t) = \beta_i \sqrt{u_i(t)}[1 - x_i(t)] - \beta_j \sqrt{u_j(t)} x_i(t)$ with $x(0) = x_0$, for $i, j = 1, 2$ and $i \neq j$, where $\beta_i > 0$ denotes the advertising effectiveness of player i. Assuming that each player maximizes a stream of discounted profits over an infinite horizon, Jarrar et al. proposed a numerical approach to compute the stationary Markov perfect Nash equilibrium.

In order to investigate the dynamic advertising competition in a duopoly setting with a leader and a follower, Breton et al. [10] developed a discrete Lanchester model with general advertising effect as, $x_{t+1} - x_t = (1 - x_t)c_l(u_{lt}) - x_t c_f(u_{ft})$, where $x_t \in [0, 1]$ denotes the market share of the leader in period $t \geq 0$; u_{it} represents the advertising expenditure of player $i \in \{l, f\}$ with l and f denoting the leader and the follower, respectively; $c_i(\cdot)$ is a positive, increasing, and concave function that measures the impact of the advertising expenditure on market share. In the sequential game, the leader decides on its advertising level $a_{lt} \geq 0$ for period t, and the follower then chooses its own advertising level

$a_{ft} \geq 0$ for period t. Assuming that the advertising efficiency functions in the dynamics are given as $c_i(u_{it}) = \beta_i\sqrt{u_{it}}$, Breton et al. proposed an algorithm for computing the feedback Stackelberg equilibrium, which is subgame perfect.

To address an advertising competition problem with a dynamic market size, Nguyen and Shi [118] modified the Lanchester model in (9) as,

$$\dot{S}_i(t) = S_i(t)\left[(b-a) - 2bQ(t)/M - \beta_i\sqrt{u_i(t)} - \beta_j\sqrt{u_j(t)}\right] + \beta_i\sqrt{u_i(t)}S(t), \text{ for } i,j = 1,2 \text{ and } i \neq j,$$

where $Q(t)$ denotes the cumulative sales of a particular new product; $S(t) \equiv \sum_{k=1}^{2} S_k(t)$ means the noncumulative industry sales at time t; a and b are the innovation and imitation coefficients, respectively; and, M is the market size. The above model was re-written as,

$$\dot{S}_i(t) = x_i(t)\dot{S}(t) + [\beta_i\sqrt{u_i(t)}S_j(t) - \beta_j\sqrt{u_j(t)}S_i(t)], \tag{10}$$

where $x_i(t) \equiv S_i(t)/S(t)$ is firm i's market share at time t. Note that, when the market is a fixed, mature one ($S(t) = M$ and $\dot{S}(t) = 0$), the sales dynamics in (10) is reduced to the model in Chintagunta and Vilcassim [22]. Nguyen and Shi attained the closed-loop Nash equilibrium for the advertising competition game where each firm maximizes its discounted profit.

4.1.2 Advertising Competition with the Pulsing Policy

In Section 3.1.1 we reviewed the V-W extensions that were developed to investigate the advertising pulsing policy. In recent years, a number of researchers have applied the Lanchester model to analyze the pulsing policy in dynamic advertising problems. Mesak and Calloway [104] developed a Lanchester extension—which is similar to Erickson's model in [34]—to evaluate the advertising policies of pulsing versus uniform spending in a duopoly setting. The instantaneous change in the sales for firm i ($i = 1,2$) at time t was described in the Lanchester manner as, $\dot{S}_i(t) = -\dot{S}_j(t) = f_i(u_i(t))S_j(t) - f_j(u_j(t))S_i(t)$, for $j = 1,2$ and $j \neq i$, where $f_i(\cdot)$ is firm i's response function such that $f_i(\cdot) > 0$ and $f_i^{'}(\cdot) > 0$. Note that $S_1(t) + S_2(t) = M$, which denotes the total sales. Setting $\dot{S}_1(t) = \dot{S}_2(t) = 0$ and solving the resulting equations for the steady-state sales rates $r_i(u_1(t), u_2(t))$, Mesak and Calloway found that $r_i(u_1(t), u_2(t)) = M - r_j(u_1(t), u_2(t)) = Mf_i(u_i(t))/[f_i(u_i(t)) + f_j(u_j(t))]$, for $j = 1,2$ and $j \neq i$. Note that the steady-state sales rate is concave in $u_i(t)$ when $f_i^{''}(\cdot) \leq 0$ and is S-shaped when $f_i^{''}(\cdot) > 0$.

Using the above, Mesak and Calloway showed that, for the rectangular or uniform advertising policies, their Lanchester extension could be re-written as,

$$\dot{S}_i(t) = -\dot{S}_j(t) = [f_i(u_i(t)) + f_j(u_j(t))][r_i(u_1(t), u_2(t)) - S_i(t)],$$

which implies that the growth (decay) parameter of each firm's sales, $f_i(u_i(t)) + f_j(u_j(t))$, depends on both firms' advertising levels; and the sales $S_1(t)$ and $S_2(t)$ can be obtained by solving the differential equation, given the initial conditions $S_i(t) = \bar{S}_i$ ($i = 1,2$) at time $t = t_0$. Using the Lanchester model in Mesak and Calloway [104], Mesak and Calloway [105] addressed the pulsing advertising competition problem by incorporating two versions of a hybrid pulsing competition subgame with a variant of the pulsing advertising competition designated by "the copycat advertising."

4.1.3 Advertising Competition with Both Brand and Generic Strategies

In practice, a firm may integrate a generic-advertising strategy for increasing the demand of its product and a brand-advertising strategy for seizing the market share from its competitors. Bass et al. [7] examined the impacts of both strategies in a duopoly setting, assuming that the total demand is $Q(t) = S_1(t) + S_2(t)$, where $S_i(t)$ $(i = 1, 2)$ is the sales of firm i and is characterized by the continuous differential function that $\dot{S}_i(t) = \dot{S}_{i,g}(t) + \dot{S}_{i,b}(t)$ with $\dot{S}_{i,g}(t)$ and $\dot{S}_{i,b}(t)$ denoting firm i's sales changes generated by the generic advertising and the brand advertising, respectively.

Suppose that firms 1 and 2 share the increase in demand generated by their generic advertising efforts. Bass et al. described firm i's sales made by the generic advertising, $S_{i,g}(t)$, using the differential equation that $\dot{S}_{i,g}(t) = \theta_i \sum_{j=1}^{2}[k_j(t)u_j(t)]$, where $\theta_i \in [0, 1]$ denotes the allocation proportion to firm i, and $u_j(t)$ and $k_j(t)$ $(j = 1, 2$ and $j \neq i)$ represent firm j's generic advertising decision and its effectiveness. In addition, Bass et al. extended Sorger's model [146] (given in Table F in the online appendix) to characterize firm i's sales generated by the brand advertising, $S_{i,b}(t)$, using the differential equation that $\dot{S}_{i,b}(t) = \beta_i \hat{u}_i(t)\sqrt{S_j(t)} - \beta_j \hat{u}_j(t)\sqrt{S_i(t)}$, where $\hat{u}_i(t)$ and β_i denote firm i's brand-advertising decision and its effectiveness. Thus, the total change in firm i's sales was attained as,

$$\dot{S}_i(t) = \beta_i \hat{u}_i(t)\sqrt{S_j(t)} - \beta_j \hat{u}_j(t)\sqrt{S_i(t)} + \theta_i \sum_{j=1}^{2}[k_j(t)u_j(t)] \text{ with } S_i(0) = \bar{S}_i. \tag{11}$$

Bass et al. derived a closed-loop Nash equilibrium for the differential game where each firm maximizes its discounted profit. By incorporating the market potential to provide an upper bound for the market demand, they also investigated the following model:

$$\dot{S}_i(t) = \beta_i \hat{u}_i(t)\sqrt{S_j(t)} - \beta_j \hat{u}_j(t)\sqrt{S_i(t)} + \theta_i \sum_{j=1}^{2}[k_j(t)u_j(t)]\sqrt{M - S_1(t) - S_2(t)} \text{ with } S_i(0) = \bar{S}_i,$$

where M is the market potential, and the term $\sqrt{M - S_1(t) - S_2(t)}$ reflects the fact that the generic advertising informs the fraction of the market that is uninformed. In [6], Bass et al. extended the model in (11) to incorporate the time-dependent allocation proportion and brand-advertising effectiveness, and obtained the closed-loop Nash equilibria for the differential games with symmetric and asymmetric competitors in a finite horizon.

4.1.4 Advertising Competition with Quality Improvement and Goodwill Accumulation

Ouardighi and Pasin [121] developed an extended Lanchester model to examine a quality-improvement and goodwill-accumulation problem in a dynamic duopoly setting. They first reviewed extended Lanchester models, which were classified into two categories. In the first category, the rate of change in each firm's market share is determined by its own current advertising effort; accordingly, the extended Lanchester model with no advertising carry-over effect is written as, $\dot{x}_i(t) = f_i(u_i(t))[1 - x_i(t)] - f_j(u_j(t))x_i(t)$ with $x_i(0) = \bar{x}_i \in [0, 1]$, for $i, j = 1, 2$ and $i \neq j$, where $f_i(\cdot)$ possesses the properties that $\partial f_i(u_i(t))/\partial u_i(t) > 0$ and $\partial^2 f_i(u_i(t))/\partial[u_i(t)]^2 \leq 0$. In the second category, each firm's market share is dependent on the firm's accumulated advertising effort; thus, the extended model is developed as $\dot{x}_i(t) = g_i(A_i(t))[1 - x_i(t)] - g_j(A_j(t))x_i(t)$ with $x_i(0) = \bar{x}_i \in [0, 1]$, for $i, j = 1, 2$ and $i \neq j$, where $A_i(t)$ is firm i's goodwill, and the function $g_i(A_i(t))$ is given such that $g_i'(\cdot) > 0$ and $g_i''(\cdot) \leq 0$. Here, the evolution of the goodwill is governed by the classic N-A model in (1).

The above two extended Lanchester categories assume that each firm can attract its competitor's customers through its current or accumulated advertising effort, but the firm cannot retain its own customers against the competitor's attraction power. Motivated by this issue, Ouardighi and Pasin added the conformance quality as a loyalty-building factor into the Lanchester model, letting the conformance quality—i.e., the proportion of non-defective items produced—represent the extent to which a product conforms to design specifications. The conformance quality of firm i is denoted by $q_i(t) \in (0,1]$ such that $\dot{q}_i(t) = \nu_i(t)[1 - q_i(t)]$ with $q_i(0) = \bar{q}_i \in (0,1]$, for $i = 1,2$, which implies that the rate of change in firm i's conformance quality depends on its own improvement effort $\nu_i(t) \geq 0$ and its own rate of defective units $[1 - q_i(t)]$. Using the term $q_i(t)$, Ouardighi and Pasin extended the Lanchester model in (9) to characterize firm i's market share as,

$$\dot{x}_i(t) = \omega_i A_i(t)[1-q_j(t)][1-x_i(t)] - \omega_j A_j(t)[1-q_i(t)]x_i(t) \text{ with } x_i(0) = \bar{x}_i \in [0,1], \text{ for } j = 1,2 \text{ and } j \neq i,$$

where $0 < \omega_i < 1$ is an adjustment coefficient reflecting firm i' attraction efficiency due to copying, media buying, and other market characteristics. Assuming that each firm's advertising cost and quality improvement cost are both quadratic, Ouardighi and Pasin obtained each firm's optimal advertising and quality improvement efforts.

4.2 The Extended Lanchester Models for the Dynamic Advertising Competition Problems in the Oligopoly Setting

Fruchter [52] extended the Lanchester model in (9) to consider a market expansion problem in the oligopoly setting where n firms use advertising to compete for customers and attract new customers from outside the oligopoly market. The saturation level (i.e., the limit of the entire industry sales) of the market at time t was assumed to be $M = \sum_{i=1}^{n} S_i(t) + \varepsilon(t)$, where $\varepsilon(t)$ denotes the market potential at time t—i.e., the difference between the saturation level of the market and the total present industry sales at time t. Firm i's ($i = 1, \ldots, n$) sales $S_i(t)$ is characterized by the extended Lanchester model as,

$$\dot{S}_i(t) = \beta_i u_i(t)[M - S_i(t)] - S_i(t) \sum_{j=1, j\neq i}^{n} \beta_j u_j(t) \text{ with } S_i(0) = \bar{S}_i. \tag{12}$$

Differentiating both sides of the saturation equation (i.e., $M = \sum_{i=1}^{n} S_i(t) + \varepsilon(t)$) with respect to t and using (12) to simplify it, Fruchter found that $\dot{\varepsilon}(t) = -\sum_{i=1}^{n} \dot{S}_i(t) = -\varepsilon(t) \sum_{i=1}^{n} \beta_i u_i(t)$ with $\varepsilon(0) = M - \sum_{i=1}^{n} \bar{S}_i$, and obtained the solution to (12) as $\varepsilon(t) = \varepsilon(0) \exp\left[-\sum_{i=1}^{n} \int_0^t \beta_i u_i(\tau) d\tau\right]$, where $\int_0^t \beta_i u_i(\tau) d\tau$ represents the cumulative effectiveness of firm i's advertising effort. Fruchter [51] investigated a multi-player differential game, adopting the extended Lanchester model in (12) but replacing the sales $S_i(t)$ with the market share $x_i(t)$. Using a Lanchester extension similar to Fruchter ([51] and [52]), Fruchter [53] investigated a growing market involving n competing firms who determine their advertising efforts to compete for customers and attract new customers.

Fruchter and Kalish [56] investigated a managerial problem regarding promotional budgeting and media allocation in a dynamic oligopoly market with fixed total industry sales, where n firms compete with L communication instruments—i.e., L different media such as advertising, sales promotions, personal selling, etc. Let $\beta_{ij} u_{ij}(t)$ denote the effectiveness of firm i's ($i = 1, \ldots, n$) promotional efforts on instrument $j \in \{1, \ldots, L\}$ at time t; here, the parameter β_{ij} is related to the effect of instrument j and to customers' perceptions and preferences on brand i. For such a problem with multi-communication instruments, Fruchter and Kalish extended the Lanchester model in (9) as $\dot{x}_i(t) = \sum_{j=1}^{L} [\beta_{ij} u_{ij}(t)] - x_i(t) \sum_{k=1}^{n} \sum_{j=1}^{L} [\beta_{kj} u_{kj}(t)]$ with $x_i(0) = \bar{x}_i$. Assuming that each firm maximizes its discounted profit, Fruchter and Kalish obtained the open-loop and the closed-loop Nash equilibria.

Naik et al. [115] addressed the problem of planning marketing mix in an oligopoly market involving multiple firms each spending on promotion and advertising to boost its brand's image. The brand managers in the firms consider both the interaction between promotion and advertising and the interaction among different competing brands. Extending the classic Lanchester model in (9) to a multiple-brand setting, Naik et al. proposed the following dynamics system: $\dot{x}_i(t) = f_i(t) - \hat{f}(t)x_i(t)$, where $x_i(t)$ is brand i's market share at time t, $f_i(t) \equiv \alpha_i u_i(t) + \beta_i v_i(t) + \gamma_i u_i(t)v_i(t)$, and $\hat{f}(t) \equiv \sum_{j=1}^{n} f_j(t)$. Note that $u_i(t)$ and $v_i(t)$ are firm i's advertising and promotion efforts, respectively; α_i and β_i characterize the impacts of advertising and promotion, respectively; and γ_i is a parameter describing the interaction effect.

The dynamics by Naik et al. [115] implies that, for each brand, the interaction between advertising and promotion affects the brand's market share; moreover, each activity (advertising or promotion) can amplify or attenuate the effectiveness of the other activity. For example, assuming a positive effect of advertising (i.e., $\alpha_i > 0$), an increase in the advertising effort $u_i(t)$ for brand i not only increases its brand share $x_i(t)$ but also affects the effectiveness of the other activity $v_i(t)$ because $\gamma_i \neq 0$. Naik et al. derived Nash equilibrium for each firm where the brand manager determines its advertising effort $u_i(t)$ and promotion plan $v_i(t)$.

Both Mesak [102] and Mesak and Darrat [107] extended the Lanchester model in (9) to investigate the advertising competition in an oligopoly setting where there are n competing films. The extended Lanchester model characterizing the sales for firm i was developed as,

$$\dot{S}_i(t) = f_i(u_i(t)) \sum_{j=1, j\neq i}^{n} S_j(t) - \left[\sum_{j=1}^{n} f_j(u_j(t))\right] S_i(t), \text{ for } i = 1, \ldots, n, \tag{13}$$

where $f_i(\cdot)$ is firm i's advertising response function such that $f_i(\cdot) \geq 0$ and $f_i'(\cdot) \geq 0$ for $u_i(t) \geq 0$; and $\sum_{j=1}^{n} S_j(t) = M$ (a constant market size). The model in (13) implies that a firm's sales rate is proportional to its advertising response function and the combined sales of its competitors. We learn from Table F (in the online appendix) that the extended Lanchester model in (13) is actually a generalized version of Little's model in [91], where the response function was specified as $f_i(u_i(t)) = \beta_i[u_i(t)]^{\alpha_i}$ with $\beta_i > 0$ and $\alpha_i > 0$. Mesak and Darrat showed that, if all the advertising response functions $f_i(u_i(t))$ are concave, then there exists a unique Nash equilibrium in which each firm chooses a uniform advertising policy in the long run.

Naik et al. [114] extended Sorger's model [146] to investigate multiple firms' optimal decisions that maximize brand awareness in a dynamic oligopoly market. Denoting by y_i the awareness of brand i at time t, Naik et al. allowed the total awareness $M = \sum_{i=1}^{n} y_i(t)$ to change over time, and developed the advertising dynamics for brand i ($i = 1, \ldots, n$) in the n-brand oligopoly market as,

$$\dot{y}_i(t) = \beta_i u_i(t)\sqrt{M - y_i(t)} - \sum_{j=1, j\neq i}^{n} \left[\beta_{ij} u_j(t)\sqrt{M - y_j(t)}\right], \tag{14}$$

where β_i is a parameter reflecting the advertising effectiveness of brand i and β_{ij} is a parameter representing the impact of the advertising effort on brand j on the awareness of brand i. Letting $\beta_{ij} = \beta_j/(n-1)$ and $M = 1$, we can find that $\sum_{i=1}^{n} y_i(t) = 1$. Naik et al. derived the closed-loop Nash equilibrium for the differential game where each firm maximizes its discounted profit.

4.3 The Extended Lanchester Models in Supply Chain Analysis

Rubel and Zaccour [133] investigated a dynamic advertising problem arising in a supply chain consisting of a manufacturer (M) and a retailer (R). The manufacturer sells its products to consumers through a distribution channel involving the retailer and an online direct channel. Each supply chain member's marketing effort at time t—$u_i(t)$, for $i \in \{M, R\}$—is assumed to keep or attract consumers to their preferred channel.

Suppose that the rate of change in the market share of the online channel depends on both the manufacturer's and the retailer's marketing efforts and on the market share of the online channel, i.e., $\dot{x}_i(t) = H(u_M(t), u_R(t), x_i(t))$. Rubel and Zaccour specified the dynamics as $\dot{x}_i(t) = [\alpha_M u_M(t) - \alpha_R u_R(t)] - \{\phi x_i(t) - \varphi[1 - x_i(t)]\}$ with $x_i(0) = \bar{x}_i$, where $\alpha_M, \alpha_R > 0$ and $0 \leq \phi, \varphi \leq 1$. Note that, in the first bracket of the specific function, the rate of change in the online market share is increasing in the manufacturer's marketing effort and decreasing in that of the retailer. The second bracket implies that consumers switch between the two channels, since ϕ and φ measure the intensities of switching by consumers in each channel to another channel. Rubel and Zaccour obtained a closed-loop Nash equilibrium for the supply chain game.

4.4 General Remarks

We present a summary of the Lanchester models appearing after 1995 in Table G (see the online appendix). Similar to Chintagunta and Vilcassim [22] and Sorger [146]—which were published before 1995, some researchers have recently developed the Lanchester models involving the dynamic equations with square effect of advertising in the duopoly setting; see Bass et al. ([6] and [7]), Jarrar et al. [69], and Nguyen and Shi [118]. In [114], Naik et al. incorporated the square form of awareness effect into their differential game models for the setting of $n \geq 3$ competing firms.

We learn from Table G that most of the Lanchester models were developed to examine the dynamic advertising problems with horizontal competition between two, or among $n \geq 3$ firms. After 1995, very few publications (e.g., Rubel and Zaccour [133]) applied the Lanchester modeling approach to the dynamic advertising problems in the supply chain setting, which should be a possible research direction. Moreover, we note that, except for Ouardighi and Pasin [121], none of the after-1995 Lanchester models has considered the product quality and the pricing decision.

5 The Diffusion Models

In a real-world advertising problem, the demand for a durable product in the current time period is often dependent on the cumulative sales of the product through a social influence (diffusion) process, in which some individuals in a market are not aware of an advertisement instantly but instead learn of the advertisement through the contact with the medium or word of mouth. Bass [4] published a seminal diffusion model for a new-product problem rather than an advertising problem. Even though the model is not related to advertising, we still begin our review of diffusion models by briefly discussing Bass's model, because many advertising-related publications appeared as its extensions.

Bass [4] assumed that, given that an individual has not purchased it before, the probability that the individual adopts a new product or technology at time t depends linearly on two factors: the first factor, denoted by α_1, is the coefficient of innovation that is positive and independent of the number of previous purchasers; and the second factor, denoted by α_2, is the coefficient of imitation that is dependent on the number of previous purchasers. Bass assumed that $f(t)$ is the likelihood of purchase at time t and $F(t) = \int_0^t f(x)dx$ is the cumulative fraction of adopters until time t. Then, Bass's diffusion model was constructed as $f(t)/[1 - F(t)] = \alpha_1 + \alpha_2 F(t)$.

Since Bass's model appeared in 1969, a number of researchers have developed and applied a variety of diffusion models to investigate advertising-related problems. Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138] provided their extensive reviews on the advertising diffusion (market growth) models, which are summarized in Table H (see the online appendix). We note that

Teng and Thompson [157] proposed a diffusion model in the general form $\dot{S}(t) = f(S(t), u(t), p(t))$; Dockner and Jøgensen's model in [29] is similar to Teng and Thompson's model but is independent of the price $p(t)$.

We next review the advertising diffusion models published after 1995, which are classified into three categories: the extended Bass models; the diffusion models with joint advertising and pricing decisions; and the diffusion models with word-of-mouth.

5.1 The Extended Bass Models

A number of researchers applied Bass's diffusion model in [4] and its extensions to analyze advertising-related problems; see a summary given by Mahajan et al. [94] in 1995. We review several major extensions of the Bass model that were published after 1995. Krishnan and Jain [87] analyzed an optimal dynamic advertising policy for new products, using the following dynamics: $\dot{S}(t) = u(t)[\alpha_1 + \alpha_2 S(t)/M][M - S(t)]$, where $S(t)$ represents the cumulative sales till time t and $\dot{S}(t)$ is the sales at time t, $u(t)$ denotes the advertising effort at time t, M is the market potential, and α_1 and α_2 are defined similarly as in Bass [4].

Jørgensen et al. [70] extended the Bass model to investigate a marketing problem of minimizing the advertising cost incurred for the sales of seats in a particular event, taking into account the effect of "word-of-mouth." The optimal-control model describing the evolution of market share was developed as $\dot{x}(t) = \beta u(t)[1-x(t)] + \gamma x(t)[1-x(t)]$ with $x(0) = 0$, where $x(t)$ represents the fraction of potential attendees who have bought a ticket by time t, $\beta > 0$ denotes the advertising efficiency, and $\gamma \geq 0$ represents the efficiency of word-of-mouth. Denote by κ the fraction of potential attendees who have seats. Since $x(t) \leq \kappa < 1$, for $t \in [0, T]$, Jørgensen et al. examined the factors that impact the organizer's advertising decisions and constructed the organizer's total cost function as $C + \kappa - x(t)$, where $C = \int_0^T (\upsilon u(t) + \varsigma [u(t)]^2 /2) dt$ is the advertising cost, and $\kappa - x(t)$ is the number of unsold seats.

Nikolopoulos and Yannacopoulos [119] extended the Bass model to discuss the optimal advertising decision on the new product diffusion in a stochastic setting, where the effect of environmental pressure on the diffusion of the product is subject to a stochastic dependence. The stochastic extension of the Bass model was given as $dx(t) = \{\alpha_1[1-x(t)] + \alpha_2 x(t)[1-x(t)]\}dt + \sigma\alpha_1[1-x(t)]d\xi(t)$, where σ is a constant and $\xi(t)$ is a Wiener process.

5.2 The Diffusion Model with Advertising and Pricing Decisions

We learn from Table H (in the online appendix) that some diffusion models published before 1995 have incorporated pricing and advertising decisions; see, e.g., Erickson [33], Mahajan and Peterson [95], Teng and Thompson [157], and Thompson and Teng [158]. Next, we focus on the review of a major after-1995 *general* diffusion model developed by Mesak [101] and Mesak and Clark [106] to investigate joint advertising and pricing decisions for a new durable product.

In Mesak [101] and Mesak and Clark [106], a monopoly firm of a new durable product was assumed to influence each customer's choice by setting its price $p(t)$, advertising effort $u(t)$, and the number of similar retail outlets $r(t)$, at time $t \in [0, T]$. The general diffusion model involving all the above factors was given as $\dot{S}(t) = f(S(t), p(t), u(t), r(t), t)$ with $S(0) = S_0 \geq 0$, where $\dot{S}(t)$ and $S(t)$ are the current sales rate and cumulative sales volume at time t.

5.3 The Diffusion Models with the Effect of Word-of-Mouth

In practice, the information of a new product may be spread by both advertising and word-of-mouth. Major relevant diffusion models published before 1995 include, e.g., Dodson and Muller [31], Kalish [79], and Muller [112]—which are briefly discussed in Table H. After 1995, some researchers still investigated the optimal advertising problem incorporating the effect of word-of-mouth. For example, using a model similar to that by Kalish [79], Swami and Dutta [149] developed optimal advertising strategies for a firm who sells a new product when either the market has not been opened up, or even if opened, the firm has not entered the market.

Kamrad et al. [80] used a stochastic extension of the diffusion model by Dodson and Muller [31]—which is given in Table H—to analyze the innovation diffusion by depicting the market at an aggregate level. The authors divided the customer population into three groups: (i) the customers who are unaware of the new product, (ii) the potential customers who are aware of the product but have not purchased it, and (iii) the customers who have purchased the product. Denote the numbers of customers in the three groups by $S_1(t)$, $S_2(t)$, and $S_3(t)$, which were assumed to evolve as a continuous-time Markov process (i.e., the Wiener process). For the ith group of customers, the instantaneous increment at time t was modeled as $dS_i(t) = f_i(S_1(t), S_2(t), S_3(t), t)dt + g_i(S_1(t), S_2(t), S_3(t), t)d\xi_i(t)$, for $i = 1, 2, 3$, where $d\xi_i(t)$ is an increment to the standard Brownian motion, $f_i(\cdot)$ and $g_i(\cdot)$ are the instantaneous drift and volatility functions.

5.4 General Remarks

We present a summary of the after-1995 diffusion models in Table I (see the online appendix), where we find that all diffusion models developed after 1995 were used to investigate the advertising problems in the monopoly setting. Note that, before 1995, Thompson and Teng [158] considered an oligopoly setting in which n firms compete in an advertising game. In recent years, some researchers have proposed stochastic diffusion models. For example, Nikolopoulos and Yannacopoulos [119] analyzed a dynamic advertising problem regarding the new product diffusion with a stochastic environmental pressure; and, Kamrad et al. [80] used a stochastic diffusion model to examine the effect of word-of-mouth on a firm's advertising policy.

6 Dynamic Advertising-Competition Models with Other Attributes

In this section, we review the after-1995 dynamic models developed for various advertising problems with other attributes (not considered in preceding sections) such as sticky price, reservation price, etc., in the presence of competition between two, or among several firms in a market.

6.1 The Dynamic Advertising-Competition Model with Sticky Price

Piga [126] developed a differential game model where two firms in a duopoly market determine their advertising efforts $u_i(t)$ and outputs $q_i(t)$ ($i = 1, 2$), assuming that the prices for their products are "sticky." The assumption implies that the change in the current price $p(t)$ is dependent on the difference between $p(t)$ and $\hat{p}(t)$, where $\hat{p}(t)$ is determined by $u_i(t)$ and $q_i(t)$ such that $\hat{p}(t) \equiv \alpha_1 + \alpha_2[u_1(t) + u_2(t)] - [q_1(t) + q_2(t)]$, where $\alpha_1 > 0$ is a constant, and α_2 denotes the effect of advertising on the price.

The adjustment process of the sticky price $p(t)$ was described by the differential function that $\dot{p}(t) = \theta[\hat{p}(t) - p(t)]$ with $p(0) = p_0$, where $\theta > 0$ denotes the speed at which the price converges to

$\hat{p}(t)$. Piga derived the feedback Nash equilibrium for the advertising competition game where each firm maximizes its own discounted profit.

6.2 The Dynamic Advertising-Competition Model with Spillover

Lambertini and Palestini [89] considered an advertising-competition game for n firms who belong to either a competitive fringe or a cartel. The firms in the competitive fringe make their advertising investments individually with no cooperation, whereas the firms in the cartel invest in a single advertising campaign with the cost symmetrically distributed among the cartel members. Suppose that there are m firms in the competitive fringe and $n-m$ firms in the cooperative cartel. At time t, the market price $p_i(t)$ for the product of firm i in the competitive fringe and the market price $p_k(t)$ for the product of firm k in the cooperative cartel are respectively given as,

$$\begin{aligned} p_i(t) &= p_i^r(t) - \beta_0 q_i(t) - \beta_1 \sum\nolimits_{j=1,\ldots,m \text{ and } j\neq i} q_j(t), \\ p_k(t) &= p_c^r(t) - \beta_0 q_k(t) - \beta_1 \sum\nolimits_{l=m+1,\ldots,n \text{ and } l\neq k} q_l(t), \end{aligned}$$

where $q_i(t)$ and $q_k(t)$ denote the sales achieved by firms i or k, respectively; $p_i^r(t)$ and $p_c^r(t)$ are the reservation prices for the product of firm i in the fringe and that for the cartel firms' products, respectively; and, β_0 and β_1 are positive constants such that $0 < \beta_1 \leq \beta_0$.

The reservation prices were assumed to evolve according to the following dynamic equations: $\dot{p}_i^r(t) = u_i(t) + \gamma \sum_{j=1,\ldots,m \text{ and } j\neq i} u_j(t) - \delta p_i^r(t)$ $(i = 1,\ldots,m)$ and $\dot{p}_c^r(t) = u_c(t) - \delta p_c^r(t)$, where $u_i(t)$ and $u_c(t)$ represent fringe firm i's and cartel firms' advertising efforts, respectively; $\delta > 0$ denotes the depreciation that affects each market size; and the parameter $\gamma \in [-1, 1]$ captures the external effects of a fringe firm's advertising on all other firms in the fringe market. Lambertini and Palestini derived the sufficient conditions for the stability of the differential oligopoly game, and analyzed the firms' profits and social welfare in Nash equilibrium.

Using dynamic models similar to Lambertini and Palestini [89], Cellini et al. [19] investigated a differential advertising game in both Cournot and Bertrand contexts. The authors found that a unique saddle-point equilibrium exists if the marginal cost of advertising is sufficiently low, and also showed that the Bertrand competition results in more intense advertising than the Cournot competition.

6.3 The Dynamic Advertising-Competition Model with Exposure Order Effects

Loginova [92] applied the theories of exposure order effects—developed in the psychology literature—to analyze the advertising competition between two firms (say, firm 1 and firm 2), who make a homogeneous product and distribute it through a same retailer. The market consists of an infinite number of identical consumers with the total mass normalized to 1. The number of times each consumer visits the retailer follows a homogenous Poisson process with the rate parameter $\lambda > 0$. Consumers form their brand preferences through the advertising by firms 1 and 2, whose advertising numbers for their brands follow homogeneous Poisson processes with rate parameters α_1 and α_2, respectively. Firm i's $(i = 1, 2)$ advertising cost was assumed as $c(\alpha_i)/r$, where $c(\alpha_i)$ possesses the properties: $c(0) = c'(0) = 0$; $c'(\alpha_i) > 0$ and $\lim_{\alpha_i\to\infty} c'(\alpha_i) = \infty$; and, $c''(\alpha_i) > 0$.

Loginova denoted by $S_i(t)$ $(i = 1, 2)$ the number of the consumers who prefer brand i at time t. The number of other consumers (i.e., the ignorant consumers), who are unaware of the product existence, is thus calculated as $S_0(t) = 1 - S_1(t) - S_2(t)$. Loginova constructed the differential equations for the primacy effect as $\dot{S}_0(t) = -(\alpha_1 + \alpha_2)S_0(t)$, $\dot{S}_1(t) = \alpha_1 S_0(t)$, and $\dot{S}_2(t) = \alpha_2 S_0(t)$; and built those for the recency effect as $\dot{S}_0(t) = -(\alpha_1 + \alpha_2)S_0(t)$, $\dot{S}_1(t) = \alpha_1 [S_0(t) + S_2(t)] - \alpha_2 S_1(t)$, and $\dot{S}_2(t) = \alpha_2[S_0(t) + S_1(t)] - \alpha_1 S_2(t)$. Using the above differential equations, Loginova derived the expected number of consumers who buy from firm i. Note that the two firms' advertising decisions are independent of time. The author then developed two firms' profit functions. Solving the resulting game (which is not a differential game) for each of three different settings, Loginova attained a traditional Nash equilibrium for two firms.

6.4 The Dynamic Advertising-Competition Model with a Dominant Firm and a Fringe Firm

Fruchter and Messinger [57] examined the advertising competition between a dominant firm and a fringe firm who may or may not be a price-taker. Let $p_d(t)$ and $u_d(t)$ denote the dominant firm's price and advertising level, respectively. The fringe firm acts as a price-taker by determining its price as $p_d(t)$, or reacts strategically by choosing a price, $p_f(t)$. For both pricing strategies, the fringe firm decides on its own advertising policy $u_f(t)$.

The rate of change in the fringe firm's sales was assumed to be dependent on both the dominant firm's and the fringe firm's current prices and advertising levels as well as the fringe firm's current sales. This was modeled in the form: $\dot{S}(t) = g(S(t), p_d(t), p_f(t), u_d(t), u_f(t))$ with $S(0) = S_0$, where $g(\cdot)$ is monotonically nondecreasing in $p_d(t)$ and $u_f(t)$, and nonincreasing in $p_f(t)$ and $u_d(t)$.

Note that $p_f(t)$ may or may not be equal to $p_d(t)$, which depends on whether or not the fringe firm is a price taker. Assuming that the total market demand is $D(p_d(t), p_f(t), u_d(t), u_f(t))$ with $\partial D(\cdot)/\partial p_d(t), \partial D(\cdot)/\partial p_f(t) < 0$ and $\partial D(\cdot)/\partial u_d(t), \partial D(\cdot)/\partial u_f(t) \geq 0$, Fruchter and Messinger calculated the demand faced by the dominant firm as $\hat{S}(t) = D(p_d(t), p_f(t), u_d(t), u_f(t)) - S(t)$. They then derived the feedback Stackelberg equilibrium for a leader-follower game where the dominant firm acts as a leader and the fringe firm acts as a follower.

6.5 General Remarks

We present Table J in the online appendix to summarize our review of the dynamic advertising models that are not discussed in Sections 2 to 5. Our summary indicates that all relevant models were proposed in the duopoly or oligopoly setting. Moreover, in addition to taking the advertising effort as a decision variable, some researchers (e.g., Fruchter and Messinger [57], Lambertini and Palestini [89], and Piga [126]) incorporated the pricing decisions into their models.

7 Empirical Studies for Dynamic Advertising Problems

A great number of researchers used the empirical data to investigate various advertising problems with the dynamic models specified in the preceding sections. Noting that very few publications (e.g., Mesak and Clark [106]) have performed empirical studies with diffusion models, we focus our review on the empirical studies with (i) the N-A models, (ii) the V-W models, and (iii) the Lanchester models.

7.1 Empirical Studies with the N-A models

Rao [130] developed a dynamic advertising-sales model in the N-A form to investigate the consequences of temporal aggregation for estimation, and also performed an empirical study to examine the analytic results. Chintagunta [20] considered the validity of the insensitivity of a firm's profit to changes in its optimal advertising level in a duopolistic market involving two firms with carryover effects. The author used the empirical data from the pharmaceutical industry to estimate the parameters of a sales response function, and then applied the estimated parameters to determine the equilibrium advertising paths for the two firms. Since the advertising effectiveness may vary over time, Naik et al. [113] incorporated a time-dependent advertising quality term into the classic N-A model in (1), and applied the advertising data of the cereal brand to estimate the model parameters, which were then compared with those in other dynamic advertising models. In addition, Bass et al. [5]—which has been reviewed in Section 2.1.4—used the empirical data collected from a major telecommunications services company in the United States to examine the interaction effects among different advertising themes (for example, price advertisements vs. product advertisements). In a recent paper concerning Internet paid search advertising, Rutz and Bucklin [134] extended the classic N-A model in (1) to capture the potential spillover from generic keyword searches to branded paid search, and also used descriptive statistics from a data set—for the paid search campaign run between Google and Yahoo search engines—to examine the positive impact of the generic search activity on the future branded search activity.

7.2 Empirical Studies with the V-W Models

Erickson [37] used the discrete V-W model developed by Deal [26] to empirically investigate the single-period advertising decisions of three large ready-to-eat cereal manufacturers in the setting of oligopolistic advertising competition. Assuming that the size of a market may vary over time, Naik et al. [114] extended Sorger's model [146]—which is an extended V-W model, as discussed in Section 3—to estimate the proposed model using market data for five car brands over time. Mesak and Ellis [108] extended the classical V-W advertising model in (6) to investigate the superiority of advertising pulsing policy over its uniform counterpart, and also employed a well-known data set (relating advertising to sales) to demonstrate their analytic results. To obtain Nash equilibrium advertising strategies for oligopolistic competitors, Erickson [40] proposed an extension of the V-W model to incorporate multiple brands for each competitor, and also estimated the model parameters using the empirical data from the carbonated soft drink market that involves three primary competitors and five primary brands. Erickson [41] developed an oligopoly model that allows the determination of feedback Nash equilibrium advertising strategies, and empirically applied the model to the triopolistic competition involving three firms in the beer industry.

7.3 Empirical Studies with the Lanchester Models

Erickson [36] generalized the Lanchester model in (9) to allow for non-linear advertising effects, and then conducted two empirical applications (involving Coca-Cola and Pepsi-Cola) to examine if the closed-loop equilibrium advertising strategies are used. Chintagunta and Vilcassim [22] also used the data for Coca-Cola and Pepsi-Cola to empirically simulate the advertising policies in both the closed- and the open-loop equilibria. Fruchter and Kalish [55] derived the closed- and the open-loop equilibria for the Lanchester model, and also found that the closed-loop equilibrium can provide a better approximation to the practice. Different from the above, Erickson [39] developed a Lanchester model in the oligopoly setting, which allows competitors to anticipate rival responses to market-share variables, and then applied the model to the cereal industry. Erickson's empirical study demonstrated that the advertising strategies based on the dynamic conjectural variations are better than the open-loop advertising strategies in explaining the advertising of the cereal competitors. Similar to Erickson [39], Naik et al. [115] considered a Lanchester model in the oligopoly setting, incorporating the interactive effects among different brands. They also conducted an empirical study to demonstrate their analytic results.

Nguyen and Shi [118] developed an extended Lanchester model to incorporate both market-share and market-size dynamics, and derived a global closed-loop Nash equilibrium. Moreover, they used a set of empirical data on sales and advertising to calculate two competing firms' optimal advertising trajectories. Breton et al. [10] applied the Lanchester model in (9) to a duopoly setting with asymmetric information and sequential play, and derived a feedback Stackelberg equilibrium. The authors also compared their analytic results with those obtained by analyzing the empirical data from the cola market.

7.4 General Remarks

Our review of recent (after-1995) empirical studies with the dynamic advertising models indicates that most of extant publications are concerned with the comparison between the analytic and the empirical results, in order to demonstrate the validity of theoretic models in characterizing or predicating the competing firms' optimal advertising policies. This would be of great help to assuring that the dynamic advertising models and analysis are mostly consistent with the reality. In addition, we find that very few publications appearing after 1995 considered the empirical investigation with the diffusion models, which may be a potential research direction in the future.

8 Summary and Concluding Remarks

In this paper, we reviewed major continuous-time dynamic optimization and differential-game models that were developed after 1995 to characterize various advertising dynamics. These models are increasingly important to investigating relevant advertising problems arising in business and economics fields, because of the fact that consumers are commonly sensitive to firms' advertising efforts and the sales achieved by firms may thus differ as a result of implementing different advertising policies. Accordingly, a number of researchers have investigated the dynamic advertising from different perspectives in the operations research, marketing, economics, and engineering fields; this is shown by the factual evidence that the after-1995 publications appeared in various academic journals belonging to the above four fields, see Table 1. Since a *very large* number of after-1995 publications are associated with dynamic advertising analysis with diverse differential function forms, we focused our attention on the review of some publications for typical mathematical models and analysis rather than all relevant publications. In fact, our review stems from the motivation to learn continuous-time differential functions for the dynamic analysis of various advertising problems.

We note from our review that dynamic advertising models published after 1995 were developed mainly to examine firms' optimal advertising decisions in *four* major system structures including (i) monopoly, (ii) duopoly, (iii) oligopoly, and (iv) supply chain systems. In addition to the advertising effort, researchers considered other decision variables or attributes, such as price, product quality, sales promotion, etc. For a summary, see Table 2, which indicates that the extant after-1995 N-A and V-W models addressed the advertising problems in all four types of system structures.

We find from Table 2 that the diffusion modeling approach has not been used to analyze the advertising problems in the duopoly, oligopoly, and supply chain structures, which may be an important area in the future. For example, we may develop a diffusion model to characterize the advertising competition between two, or among three or more retailers. For such a horizontal competitive structure (marketing channel), we may find the retailers' advertising policies in Nash equilibrium and also

Field	Journal	Number of Representative Publications
Operations Research (44 publications)	Annals of Operations Research	**1** ([1])
	Computers & Operations Research	**1** ([101])
	European Journal of Operational Research	**17** ([11], [41], [42], [56], [59], [74], [80], [81], [86], [88], [96], [98], [104], [108], [116], [121], and [162])
	Journal of Industrial and Management Optimization	**1** ([6])
	Journal of Optimization Theory and Applications	**11** ([15], [18], [54], [62], [73], [75], [97], [109], [140], [144], and [160])
	Journal of the Operational Research Society	**2** ([103] and [107])
	Mathematical Methods of Operations Research	**1** ([14])
	Management Science	**6** ([10], [47], [69], [87], [114], and [118])
	Nonlinear Analysis: Real World Applications	**1** ([119])
	Operations Research	**1** ([40])
	Operations Research Letters	**1** ([65])
	Production and Operations Management	**1** ([66])
Marketing (5 publications)	Journal of Retailing	**1** ([71])
	Marketing Science	**4** ([21], [5], [7], and [115])
Economics (8 publications)	Annals of the International Society of Dynamic Games	**1** ([133])
	International Game Theory Review	**2** ([72] and [120])
	International Journal of Industrial Organization	**1** ([126])
	Journal of Economic Behavior and Organization	**1** ([92])
	Journal of Economic Dynamics and Control	**2** ([57] and [76])
	Regional Science and Urban Economics	**1** ([9])
Engineering (13 publications)	Automatica	**8** ([12], [63], [70], [77], [99], [128], [129], and [164])
	Mathematical and Computer Modelling	**1** ([13])
	Optimal Control Applications and Methods	**4** ([52], [89], [106], and [163])

Table 1: The distribution of the after-1995 relevant publications in academic journals. Note that, in each field, the journals are listed in alphabetical order.

		N-A Models	V-W Models	Lanchester Models	Diffusion Models	Other Models
System Structure	Monopoly	•	•		•	
	Duopoly	•	•	•		•
	Oligopoly	•	•	•	•	•
	Supply Chain	•	•	•		
Decision Variable	Advertising	•	•	•	•	•
	Price	•	•		•	•
	Quality	•	•	•		
	Sales Promotion			•		

Table 2: The system structures and decision variables considered in five categories of the extant after-1995 dynamic advertising models. The solid point "·" means that a system structure or a decision variable has been considered in a corresponding category.

investigate the advertising competition at the retailing level if the retailers compete with, e.g., a new durable product. Moreover, we may consider a two-echelon supply chain where a manufacturer subsidizes the advertising for multiple retailers. For such a supply chain, we could determine the manufacturer's and the retailers' equilibrium advertising policies.

Our review also reveals that the majority of extant models for the dynamic advertising problems in supply chain operations have been constructed to determine firms' advertising decisions in a *single* supply chain. Very few publications have considered the dynamic advertising problems in a market with two or more competing supply chains. For example, in [99], Martín-Herrán et al. analyzed a dynamic advertising problem involving two competing manufacturers and a single retailer. As a possible future research topic, we may analyze the competition between two supply chains, each consisting of a manufacturer and a retailer. The two manufacturers produce and distribute homogenous or substitutable products to consumers in a market through their exclusive retailers, which may compete for the market share with their advertising efforts, similar to Rubel and Zaccour [133]. Moreover, the two manufacturers may also invest in the advertising of their brands via a media accessible to consumers in a mass market; and similar to Jørgensen et al. [71], each manufacturer may decide on whether or not to entice its downstream retailer with a subsidy to exert a larger advertising effort. Accordingly, for the setting of two competing supply chains, it would be interesting to investigate a sequential game decision problem as follows: First, two manufacturers "simultaneously" determine their advertising investment levels and subsidies to their retailers (in the case of supporting their retailers); and then, two retailers "concurrently" make their advertising decisions. Solving the above two-stage game, we would find the answer to the following important question: Should a manufacturer subsidize the advertising investment at the retailing level in its supply chain? In the above supply chain structure, we may also consider the wholesale and the retail pricing decisions for the manufacturers and the retailers, respectively.

We learn from our review that, except for the models by Giovanni [59] (N-A), Nair and Narasimhan [116] (N-A), Ouardighi et al. [120] (V-W), and Ouardighi and Pasin [121] (Lanchester), very few publications appearing after 1995 have extended the N-A model, V-W model, Lanchester model, or diffusion model to investigate the dynamic advertising problems with the product quality as a decision variable. In addition, the pricing decision has not been incorporated into any extant extended Lanchester model. Our review also indicates that, except for the models by Fruchter and Kalish [56] and Naik et al. [115], other extant dynamic optimization and differential-game models have not considered the sales promotion decision in the dynamic advertising problems.

In addition to the above discussion, we find that most publications have assumed general or specific differential functions to address a variety of dynamic advertising problems for an aggregate market, whereas very few publications have investigated consumer behavior using the utility-based approach. In the future, it would be interesting to apply the utility theory to construct and analyze dynamic advertising models. To help understand this research direction, we next briefly discuss three major relevant publications appearing in the past decade. As discussed in Section 2.1.4, Lambertini [88] used a utility-based method to derive the demand faced by a firm and then characterize the N-A dynamics of advertising investment in a spatial monopoly. We also recall from Section 2.2 that Bertuzzi and Lambertini [9] recently built a differential game in the N-A form to examine the competition between two firms who compete with product differentiation and advertising efforts. In [9], each consumer's utility on a product is dependent on his or her transportation cost and the two firms' pricing decisions; and the total market demand is only dependent on the two firms' advertising efforts. Moreover, we find from Section 6 that Lambertini and Palestini [89] examined an advertising-competition game in an oligopoly setting, where the demand functions were derived by using the utility function developed by Spence [147]. The above discussion indicates that, in order to apply the utility theory for the dynamic advertising analysis, we should construct consumers' time-dependent utility functions that also depend on a firm's or multiple firms' decision variables, and then analyze the utility functions to derive the demand faced by each firm, which can be used to develop the firm's profit or cost function.

We note that a few researchers recently used the consumer utility function to propose some static models for investigating a firm's optimal advertising policy or the horizontal advertising competition between two, or among three or more firms. For example, Kind et al. [83] analyzed how competitive forces influence the way in which media firms (e.g., TV channels) raise their revenues, deriving the demand function for each media firm by using Shubik and Levitan's utility function in [143]. Godes et al. [60] investigated the impact of content and advertising competition in an oligopoly setting, where they adopted the consumer utility function in Vives [161] to derive the demand functions for multiple firms. We also observe that, in recent years, some researchers have used the Hotelling model—which is a consumer utility-based choice model (see Hotelling [68])—to estimate the demand. For example, Banerjee and Bandyopadhyay [3] developed a multi-stage game model to investigate the advertising and price competition between two firms, and Dukes and Gal-Or [32] obtained the conditions under which exclusive advertising contracts are beneficial to both the advertisers and the media outlets. In addition, Gal-Or and Gal-Or [58] investigated whether or not a single media content distributor should utilize the customized advertising to implement a monopoly pricing strategy when the distributor delivers advertising messages for multiple competing brands.

In the future, we may extend the above utility approaches used in static advertising problems by incorporating a proper time factor, and construct the dynamic advertising models in the N-A, the V-W, the Lanchester, and the diffusion forms. This would be a possible direction for the future research.

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Online Appendix
"Recent Developments in Dynamic Advertising Research"

J. Huang, M. Leng, L. Liang

Review Paper	System Dynamics	Representative References
Sethi [138]	$S(t) = S(p(t), A(t), Z(t))$ with $\dot{A}(t) = u(t) - \delta A(t)$.	Nerlove and Arrow [117]
	$\dot{A}_i(t) = \int_{-\infty}^{t} u_i(t)w_i(t-\tau)d\tau$ with $\int_{-\infty}^{t} w_i(t-\tau)d\tau = 1$, for $i = 1, \ldots, n$, where $w_i(t-\tau)$ is the weight of the expenditure $u_i(t)$ for the calculation of goodwill.	Bensoussan et al. [8]
	$dA(t) = [u(t) - \delta A(t)]dt + [u(t) + \delta A(t)]^{1/2}d\xi(t)$, where $d\xi(t)$ is a standard Wiener process.	Tapiero [151]
	$\dot{P}(A(t), t) = \delta(A(t)+1)P(A(t)+1, t) - [\delta(A(t)) + \rho(A(t), u(t), M)]P(A(t), t)$ $+\rho(A(t), u(t), M)P(A(t)-1, t)$, where $P(A(t), t)$ is the probability of goodwill being $A(t)$ at time t and M is the upper bound on the amount of goodwill.	Tapiero [152]
Feichtinger, Hartl, and Sethi [44]	$\dot{A}_i(t) = \int_{-\infty}^{t} w(t-\tau)g(u(\tau))d\tau - \delta A(t)$, for $i = 1, \ldots, n$, where $w(\cdot)$ is the density function and $g(u(\tau))$ is the advertising effectiveness function (a production function for goodwill).	Pauwels [124]
	$\dot{A}(t) = u(t) - \delta A(t)$ and $\dot{V}(t) = u(t) + \delta A(t) - 2\delta V(t)$, where $A(t)$ and $V(t)$ are the mean value and the variance of the goodwill, respectively.	Tapiero [153]
	$\dot{Q} = k(q - Q) + g(u(t))$, where q and Q are product quality and consumers' prior quality expectation, respectively; and $k \in (0, 1)$ is the speed of learning.	Kotowitz and Mathewson ([84], [85])
	$\dot{S}(t) = u(t) - \delta(q)S(t)$.	Tapiero [155]
	$\dot{x} = g(u(t))(1-x) - \delta(u(t), q)x$, where x denotes the market potential.	Ringbeck [131]
	$\dot{A}(t) = q - Q(u(t), A(t))$.	Conrad [25]
	$\dot{P}_i(t) = \lambda u(t)P_{i-1}(t) - [\lambda u(t) + i\mu]P_i(t) + (i+1)\mu P_{i+1}(t)$ with $\dot{P}_0(t) = -\lambda u(t)P_0(t)$ $+\mu P_1(t)$ (for $i \geq 1$), where $P_i(t)$ is the probability that there are i units of goodwill at time t.	Rishel [132]
	$S(t) = p(t)^{-\eta}A(t)^{\omega}\phi(R(t))$ with $\dot{A}(t) = u(t) - \delta A(t)$ and $\dot{R}(t) = h(p^0/p(t))S(t) - \gamma R(t)$, where $R(t)$ denotes the firm's reputation, p^0 means the reference price, and γ represents the rate of depreciation of reputation. Moreover, $h(p^0/p(t)) < (\geq)0$ when $p^0 < (\geq)p(t)$.	Spremann [148]
	$S(t) = \beta_0 + \beta_1 A(t) + z(t)$ with $dA(t) = [u(t) - \delta A(t)]dt + \sigma_2 d\xi(t)$, where β_0 and β_1 are constants and $z(t)$ is a stochastic process satisfying $dz(t) = -\rho z(t)dt + \sigma_1 d\xi(t)$.	Rao [130]
	$\dot{S}(t) = g(A(t)) - \alpha S(t) + \omega(A(t) - \bar{A}(t))$ with $\dot{A}(t) = u(t) - \delta A(t)$ and $d\bar{A}(t)/dt$ $= \zeta[A(t) - \bar{A}(t)]$, where α is the proportion of consumers switching to other brand per unit time, $\omega(A(t) - \bar{A}(t))$ is the extra effect gained when $A(t)$ is above the adaption level $\bar{A}(t)$, and ζ is the relative weight of more recent levels of advertising capital.	Luhmer et al. [93]
	$A(t) = \int_{-\infty}^{t} g(u(s), A(s))\tilde{l}(t-s)ds$ where $\tilde{l}(t) = \int_{-\infty}^{t} l(t-\tau)w(\tau)d\tau$ with $\tilde{l}(-\infty) = 0$, $\tilde{l}(\infty) = 1$, and $\tilde{l}'(t) \geq 0$.	Feichtinger et al. [44]
Erickson [38]	$\dot{S}(t) = -mS(t) + \sum_{i=1}^{n} f(u_i(t))$ with $P(S_i(t)\|S(t), t) = \exp(-SP_i)(SP_i)^{S_i(t)}/S_i(t)!$, where $S(t)$ is the Poisson-distributed industry sales at time t, $m > 0$ is a constant, $f(u_i(t))$ is a function of firm i's ($i = 1, \ldots, n$) advertising expenditure, $P(S_i(t)\|S(t), t)$ is the probability that firm i's sales conditional on $S(t)$ at time t are $S_i(t)$, and SP_i is the mean value of $S_i(t)$.	Tapiero [154]
	$S_i(t) = [A_i(t)]^{\alpha}/\sum_{j=1}^{n}[A_j(t)]^{\alpha}$ with $\dot{A}(t) = u(t) - \delta A(t)$, where $\alpha \geq 0$ is a constant and $A(t) = \sum_{j=1}^{n} A_j(t)$ with $A_i(t)$ (for $i = 1, \ldots, n$) denoting firm i's goodwill stock at time i.	Fershtman [48]
	$S_i(t) = [A_i(t)]^{\alpha_i}[A_j(t)]^{-\beta_i}b_i(p_i, p_j)$ with $\dot{A}_i(t) = u_i(t) - \delta_i A_i(t)$, for $i, j = 1, 2$ and $i \neq j$.	Fershtman et al. [49]
	$S_i(t) = \alpha_i A_i(t) - \beta_i A_j(t) - \gamma_i[A_i(t)]^2 + \theta_i[A_j(t)]^2 + \phi_i A_i(t)A_j(t)$ with $\dot{A}_i(t) = \sqrt{u_i(t)} - \delta A_i(t)$, for $i, j = 1, 2$ and $i \neq j$.	Chintagunta [20]

Table A: A summary of the Nerlove-Arrow advertising model and major extensions surveyed by Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138]. Note that the review papers and the references (for each review paper) are both listed in chronological order.

References	System Dynamics	Number of Players	Solution/ Equilibrium	Time Horizon
Mosca and Viscolani [109]	$\dot{A}(t) = f(u(t)) - \delta A(t)$, where $f^{'}(\cdot) > 0$ and $f^{''}(\cdot) < 0$.	1 (Monopoly)	OS	finite
Buratto and Viscolani ([13], [14])	$\dot{A}(t) = [u(t)]^{\alpha} - \delta A(t)$, where $\alpha \in (0, 1]$.	1 (Monopoly)	OS	finite
Buratto et al. [11]	$\dot{A}(t, a) \equiv \partial A(t, a)/\partial t = u(t, a) - \delta(a)A(t, a)$, with $A(0, a) = \alpha(a) \geq 0$ and $\delta(a) > 0$ for $a \in \Omega$.	1 (Monopoly)	OS	finite
Buratto et al. [12]	$\partial A(t, a)/\partial t = \gamma(a)\varphi(u(t)) - \delta(a)A(t, a)$, with $A(0, a) = \alpha(a) \geq 0$, $\sum_{a\in\Omega} \gamma(a) = 1$, and $\varphi(u(t)) \geq 0$.	1 (Monopoly)	OS	finite
Marinelli and Savin [97]	$\partial A(t, \hat{a})/\partial t = -\delta A(t, \hat{a}) + \triangle_{\hat{a}} A(t, \hat{a}) + b(\hat{a})u(t, \hat{a})$, where $A : [0, T] \times \Xi \to \mathbb{R}$ with $\Xi \subset \mathbb{R}$ is the goodwill "density," and $\hat{a}$ is a spatial coordinate.	1 (Monopoly)	OS	finite
Fruchter [54]	$\dot{A}(t) = kp(t) + \rho u(t) - \delta A(t)$ with $A(0) = A_0$.	1 (Monopoly)	OS	infinite
Erickson [42]	$S(t) = \alpha - \beta p(t) + \gamma A(t)$, where $\dot{A}(t) = u(t) - \delta A(t)$.	1 (Monopoly)	FNE, FSE	infinite
Raman [129]	$dA(t) = [\beta u(t) - \delta A(t)]\, dt + \sigma d\omega(t)$ with $A(0) = A_0$.	1 (Monopoly)	OS	finite
Marinelli [96]	$dA(t) = [u(t) - \rho A(t)]dt + \sigma(A(t), u(t))d\omega(t)$ with $A(0) = A_0$.	1 (Monopoly)	OS	finite
Gozzi et al. [62]	$dA(t) = [\lambda_0 A(t) + \int_{-\gamma}^{0} \lambda_1(\theta)A(t + \theta)d\theta + \hat{\lambda}_0 u(t)+$ $\int_{-\gamma}^{0} \hat{\lambda}_1(\theta)u(t + \theta)d\theta]dt + \sigma d\omega(t)$, for $t_0 \leq t \leq T$, $A(t_0) = A_0$, $A(t_0 + \theta) = A_1(\theta) \geq 0$, $u(t_0 + \theta) = \delta(\theta)$, and $\theta \in [-\gamma, 0]$.	1 (Monopoly)	OS	finite
Lambertini [88]	$\dot{p}_r(t) = \delta_1 \sqrt{u(t)} - \delta_2 p_r(t)$, for $\delta_1 > 0$, where $p_r(t)$ is the consumers' reservation price.	1 (Monopoly)	OS	infinite
Grosset and Viscolani [63]	$\dot{A}(t) = \gamma_1 u(t) - \gamma_2 - \delta A(t)$ with $A(0) = A_0$.	1 (Monopoly)	OS	infinite
Buratto and Zaccour [15]	$\dot{S}(t) = \alpha_L A_L(t) + \alpha_l A_l(t)$ with $S(0) = 0$, where $\dot{A}_L(t) = \eta_L u_L(t) - \delta A_L(t)$ with $A_L(0) = \bar{A}_L$, and $\dot{A}_l(t) = \eta_l u_l(t) + \beta_L A_L(t) - \delta A_l(t)$ with $A_l(0) = \bar{A}_l$.	2 (Duopoly)	OLSE	finite
Bass et al. [5]	$\dot{A}(t) = \sum_{i=1}^{k} \psi_i(t) \left[g_i(u_i(t)) + \lambda_i \sum_{j=1, j\neq i}^{k} h(u_i(t), u_j(t))\right]$ $-\delta A(t)$, where $\dot{\psi}_i(t) = -a(u_i(t))\psi_i(t) + \delta[1 - I(u_i(t))][1 - \psi_i(t)]$.	1 (Monopoly)	—	—
Weber [163]	$\dot{x}(t) = [1 - x(t)]A(t) - \beta x(t)$, where $\alpha, \beta > 0$ and $k \in (0, 1)$, and $\dot{A}(t) = [u(t)]^k - \alpha A(t)$.	1 (Monopoly)	OS	infinite
Nair and Narasimhan [116]	$S_i(t) = \max\{0, [\alpha - \beta p_i(t)][g_1 A_i(t) - g_2[A_i(t)]^2/2]\}$, for $i, j = 1, 2$ and $j \neq i$, where $\dot{A}_i(t) = q_i(t) - \theta_q q_j(t) + u_i(t) - \theta_q u_j(t) - \delta A_i(t)$.	2 (Duopoly)	OLNE	infinite
Bertuzzi and Lambertini [9]	$S_1(t) = M(t)\bar{l}(t)$, $S_2(t) = M(t) - S_1(t)$, where $\dot{M}(t) = \alpha[u_1(t) + u_2(t)] - \delta M(t)$, and $\bar{l}(t) \equiv [p_2(t) - p_1(t) + k(y_1(t) + y_2(t))]/(2k)$.	2 (Duopoly)	FNE	infinite
Cellini and Lambertini [18]	$p(t) = [p_R(t) - S(t)]^{1/\alpha}$, where $S(t) = \sum_{i=1}^{n} S_i(t)$, $\alpha \in (0, \infty)$, and $\dot{p}_R(t) = \sum_{i=1}^{n} u_i(t) - \delta p_R(t)$ with $p_R(0) = \bar{p}_R$.	$n \geq 3$ (Oligopoly)	OLNE, FNE	infinite
Viscolani and Zaccour [160]	$S_i(t) = \beta A_i(t)$, for $i, j = 1, 2$ and $j \neq i$, where $\dot{A}_i(t) = \gamma_i u_i(t)$ $-\eta_{ij}\gamma_j u_j(t) - \delta A_i(t)$ with $A_i(0) = \bar{A}_i$ and $\gamma_i \geq 0$.	2 (Duopoly)	OLNE	infinite

Table B: A summary of the after-1995 N-A models that are reviewed for the dynamic advertising problems in the monopoly, duopoly, and oligopoly settings. Note that we summarize the references in the sequence in which they are reviewed in Section 2. Moreover, in the column "Solution/equilibrium," (i) OS, (ii) FNE, (iii) FSE, (iv) OLSE, and (v) OLNE represent (i) optimal solution, (ii) feedback Nash equilibrium, (iii) feedback Stackelberg equilibrium, (iv) open-loop Stackelberg equilibrium, and (v) open-loop Nash equilibrium, respectively.

References	System Dynamics	Number of Players	Equilibrium	Time Horizon
Chintagunta and Jain [21]	$S(t) = S(A_M(t), A_R(t))$, where $\dot{A}_i(t) = u_i(t) - \delta A_i(t)$ with $A_i(0) = \bar{A}_i$, for $i = M, R$.	2 (1M+1R)	OLNE, FNE	infinite
Jørgensen et al. [71]	$S(u_S(t), v_S(t), A(t)) = [\alpha_M u_S(t) + \alpha_R v_S(t)]\sqrt{A(t)}$, where $\dot{A}(t) = \lambda_M u_L(t) + \lambda_R v_L(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	FNE	infinite
Jørgensen et al. [73]	$S(u(t), A(t)) = \beta u(t) + \lambda A(t) - \gamma [A(t)]^2$, where $\dot{A}(t) \equiv dA(t)/dt = u(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	FNE, FSE	infinite
Jørgensen et al. [74]	$S(u_R(t), A(t)) = \gamma u_R(t) + \theta A(t)$, where $\dot{A}(t) = \alpha u_M(t) - \beta u_R(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	FNE, FSE	infinite
Jørgensen and Zaccour [77]	$S(t) = \gamma u_R(t) + \theta A(t)$, where $\dot{A}(t) = u_M(t) - \beta K(t)$, $\dot{K}(t) = n \times u_R(t) - \mu K(t)$, $A(0) = A_0$, and $K(0) = K_0$.	$n + 1$ (1M+nR)	MNE	finite
Jørgensen and Zaccour [75]	$S(t) = [\alpha - \beta p(t)]\{g_1 A(t) - g_2 [A(t)]^2 /2\}$, where $\dot{A}(t) = u_M(t) + u_R(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	MNE	infinite
Jørgensen et al. [72]	$S(t) = u_R(t)[\alpha - \beta p(t)]\sqrt{A(t)}$, where α, $\beta > 0$, and $\dot{A}(t) = u_M(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	FNE, FSE	infinite
Jørgensen and Zaccour [76]	$S(t) = [\alpha - \beta p(t)]\{g_1 A(t) - g_2 [A(t)]^2 /2\}$, where $\dot{A}(t) = \lambda_M u_M(t) + \lambda_R u_R(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	IE	finite
Zaccour [164]	$S(t) = \mu + A(t) - \alpha p(t)$, where μ, $\alpha > 0$, and $\dot{A}(t) = u_M(t) + u_R(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	FNE	infinite
Amrouche et al. [1]	$S_i(t) = \beta_i + A_i(t) - p_i(t) + \psi p_j(t)$, where $\psi \in [0, 1)$, and $\dot{A}_i(t) = u_i(t) - \delta A_i(t) - k_i u_j(t)$ with $A_i(0) = A_0$, for $i, j \in \{NB, SB\}$ and $j \neq i$.	2 (1M+1R)	FSE	infinite
Karray and Martín-Herrán [81]	$S_i(t) = b_i + \psi_i A_i(t) + \theta_i A_j(t) - p_i(t) + \alpha[p_j(t) - p_i(t)]$, for $i, j = NB, SB$ and $i \neq j$, where $\dot{A}_i(t) = \delta\sqrt{u_i(t)} - \lambda A_i(t)$ with $A_i(0) = \bar{A}_i$,	2 (1M+1R)	FNE, FSE	infinite
Giovanni [59]	$S(t) = \alpha - \beta\eta w(t) + \theta A(t)$, where $\varepsilon, \gamma, \delta, \alpha, \beta, \theta > 0$, $\eta > 1$, and $\dot{A}(t) = \varepsilon u(t) + \gamma q(t) - \delta A(t)$ with $A(0) = A_0$.	2 (1M+1R)	FSE	infinite
Sigué and Chintagunta [144]	$S_i(t) = \{\alpha u_i(t) + \beta[u_i(t) - u_j(t)]\}\sqrt{A(t)}$, for $i, j = 1, 2$, and $i \neq j$, where $\dot{A}(t) = v_1(t) + v_2(t) - \delta A(t)$ with $A(0) = A_0$.	3 (1Fr+2Fe)	FSE	infinite
Martín-Herrán et al. [98]	$S_i(t) = \alpha + \theta A(t) - p_i(t) + \gamma p_j(t) + \lambda u_i(t) + \mu u_j(t)$, for $i, j = 1, 2$ and $i \neq j$, where $\alpha, \theta > 0$; $0 \leq \gamma < 1$, $\lambda \geq 0$; $\lvert\mu\rvert \leq \lambda < 1$; and $\dot{A}(t) = k u_M(t) - \delta A(t)$ with $A(0) = A_0$.	3 (1Fr+2Fe)	FSE	infinite
Martín-Herrán et al. [99]	$S_i(t) = \beta_i(t)[a_i A_i(t) - b_i\beta_i(t)/2]$, for $i, j = 1, 2$ and $j \neq i$, where $a_i, b_i > 0$, and $\dot{A}_i(t) = \alpha_i u_i(t) - \delta A_i(t)$ with $A_i(0) = \bar{A}_i$.	3 (2M+1R)	OLSE	infinite

Table C: A summary of the after-1995 N-A extensions for the dynamic advertising problems in the supply chain setting. Note that we summarize the references in the sequence in which they are reviewed in Section 2. In the column "Number of players," M, R, Fr, and Fe represent a manufacturer, a retailer, a franchisor, and a franchisee, respectively; and in the column titled "Equilibrium," MNE and IE represent the Markovian Nash equilibrium and the incentive equilibrium, respectively, and other abbreviations are defined as in the caption of Table B.

Review Paper	System Dynamics	Representative References
Sethi [138]	$\dot{S}(t) = ru(t)[M - S(t)]/M - \lambda S(t)$.	Vidale and Wolfe [159]
	$\dot{S}(t) = g(S(t), u(t), t)$ with $\partial g(\cdot)/\partial S(t), \partial g(\cdot)/\partial u(t) > 0$, and $\partial^2 g(\cdot)/\partial [u(t)]^2 \leq 0$.	Sasieni [135]
	$S(t+1) - S(t) = r[\alpha - S(t)] + \beta u(t)$ with $S(1) = S_1$, where α and β are two positive constants, and r is a constant rate at which the goodwill depreciates.	Sethi et al. [142]
	$\dot{x}(t) = \rho u(t)[1 - x(t)] - kx(t)$ with $x(0) = x_0$, where $x(t) \equiv S(t)/M$ means the market share (i.e., the ratio of the sales $S(t)$ to the market potential M), $\rho \equiv r/M$, and $k \equiv \lambda + \dot{M}/M$. Note that the notations $S(t)$, M, r, and λ were used in the basic V-W model in [159].	Sethi [136]
	$\dot{S}(t) = r \ln[u(t)] - \lambda S(t)$.	Sethi [137]
	$dx(t) = \{\rho u(t)[1 - x(t)] - kx(t)\}dt + \{\rho u(t)[1 - x(t)] - kx(t)\}^{1/2} d\xi(t)$, where $\xi(t)$ is a standard Wiener process.	Tapiero [150]
	$S(t+1) - S(t) = r \sum_{l=0}^{t} \psi_{tl} u(l) - kS(t)$, where ψ_{tl} is the weight of $u(l)$ at time t.	Burdet and Sethi [16]
Feichtinger, Hartl, and Sethi [44]	$\dot{x}(t) = \rho u(t) x(t)[1 - x(t)] - kx(t)$.	Ozga [122]
	$dx(t) = g(x(t), u(t))dt + \sigma(x(t), u(t)) d\xi(t)$, where $g(\cdot)$ and $[\sigma(\cdot)]^2$ are the drift and the instantaneous variance of the $x(t)$ process.	Fleming and Rishel [50]
	$\dot{x}_i(t) = \rho(t - T_i, I_i) u_i(t)[1 - x_i(t)] - kx_i(t)$ with $t \in [T_i, T_{i+1}]$, for $i = 1, 2, \ldots, m-1$, where $\rho(t - T_i, I_i)$ is a stochastic process whose distribution depends on the amount of investment I_i.	Pekelman and Sethi [125]
	The continuous-time differential function is the same as that in Fleming and Rishel [50]; but, $g(x(t), u(t)) = \rho u(t)\sqrt{1 - x(t)} - kx(t)$.	Sethi [139]
	$\dot{y}(t) = f(u(t))[\bar{y} - y(t)] - \delta y(t)$, where $y(t)$ is the awareness created by advertising, $\bar{y}$ is complete awareness of the entire population, $f(\cdot)$ is an S-shaped (non-concave) function, δ is the forgetting rate.	Mahajan and Muller [110]
	$\dot{y}(t) = u(t)[\bar{y} - y(t)] - \delta y(t)$.	Hahn and Hyun [64]
	$\dot{x}(t) = g(x(t), z(t))$ with $\dot{z}(t) = \zeta[u(t) - z(t)]$, where $z(t)$ denotes the filtered variable which the consumer perceives instead of $u(t)$.	Feinberg [46]
Erickson [38]	$\dot{S}_i(t) = r_i u_i(t)\{1 - S_i(t)/[S_1(t) + S_2(t)]\} - \lambda_i S_i(t)$, for $i = 1, 2$.	Mukundan and Elsner [111]
	$\dot{S}_i(t) = u_i(t) - c_i[u_i(t)]^2/2 - \lambda_i S_i(t) - \beta_i u_{3-i}(t) S_i(t)$, for $i = 1, 2$.	Leitmann and Schmitendorf [90]
	$\dot{S}_i(t) = r_i u_i(t)\{1 - [M - S_1(t) + S_2(t)]/M\} - \lambda_i S_i(t)$, for $i = 1, 2$.	Deal [26]
	$\dot{S}_i(t) = r_i u_i(t)[1 - S_1(t) - S_2(t)] - \lambda_i S_i(t) + \beta_i[u_i(t) - u_{3-i}(t)][S_1(t) + S_2(t)]$, for $i, j = 1, 2$ and $i \neq j$.	Deal et al. [27]
	$\dot{S}_i(t) = g(u_i(t)) - \lambda_i S_i - \hat{g}_i(u_{3-i}(t)) S_i(t)$, for $i = 1, 2$.	Feichtinger [43]
	$\dot{S}_i(t) = f_i(u_i(t))[M - \sum_{j=1}^{n} S_j(t)] - \lambda_i S_i(t)$, for $i = 1, \ldots, n$.	Erickson [38]

Table D: A summary of the Vidale-Wolfe advertising model and major extensions surveyed by Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138]. Note that the review papers and the references (for each review paper) are both listed in chronological order.

References	System Dynamics	Number of Players	Solution/ Equilibrium	Time Horizon
Mesak [103]	$\dot{S}(t) = f(u(t))[M - S(t)]/M - \lambda S(t)$.	1 (Monopoly)	OS	finite
Mesak and Ellis [108]	$\dot{S}(t) = f(u(t))[M(u(t)) - S(t)]/M(u(t)) - \lambda S(t)$, with $f(u(t)) > 0$, $M(u(t)) > 0$, $\partial f(u(t))/\partial u(t) \geq 0$, $\partial M(u(t))/\partial u(t) \geq 0$, $\partial^2 f(u(t))/\partial u^2(t) \leq 0$, and $\partial^2 M(u(t))/\partial u^2(t) \leq 0$.	1 (Monopoly)	OS	finite
Feinberg [47]	$\dot{S}(t) = f(u(t))a(u(t)) - b(u(t))$, where $f(u(t))$ is $S-$shaped.	1 (Monopoly)	OS	infinite
Prasad and Sethi [128]	$dx(t) = [U(u(x(t)), v(x(t)))\sqrt{1-x(t)} - kx(t)]dt + \sigma(x(t))d\xi(t)$, where $U(u(x(t)), v(x(t))) \equiv \rho_u u(x(t)) + \rho_v v(x(t)) + \hat{k}\sqrt{u(x(t))v(x(t))}$, with $\rho_u > 0$, $\rho_v > 0$, and $\hat{k} \geq 0$.	1 (Monopoly)	OS	infinite
Sethi et al. [140]	$\dot{S}(t) = ru(t)D(p(t))\sqrt{1-S(t)} - \lambda S(t)$ with $D^{'}(p(t)) < 0$.	1 (Monopoly)	OS	infinite
Prasad and Sethi [127]	$dx_i(t) = \{\rho_i u_i(x_i(t))\sqrt{1-x_i(t)} - \rho_j u_j(x_i(t))\sqrt{x_i(t)} - k[2x_i(t) - 1]\}dt + \sigma(x_i(t))d\xi(t)$, for $i = 1, 2$ and $i \neq j$.	2 (duopoly)	FNE	infinite
Erickson [41]	$\dot{S}_i = a_i\beta_i\sqrt{M - \sum_{j=1}^{n} S_j(t)} - \rho_i S_i$, for $i = 1, 2, \ldots, n$.	$n \geq 3$ (oligopoly)	FNE	infinite
Erickson [40]	$\dot{S}_{ij}(t) = r_{ij}u_{ij}(t)\sqrt{M - \sum_{k=1}^{n}\sum_{l=1}^{J_k} S_{kl}(t)} - \lambda_{ij}S_{ij}(t)$, for $i = 1, 2, \ldots, n$ and $j = 1, \ldots, J_i$.	$n \geq 3$ (oligopoly)	FNE	infinite
Krishnamoorthy et al. [86]	$\dot{S}_i(t) = r_i u_i(t)D_i(p_i(t))\sqrt{M - S_i(t) - S_j(t)}$, for $i, j = 1, 2$ and $i \neq j$.	2 (duopoly)	FNE	infinite
He et al. [66]	$dx(t) = (\rho u(t)\sqrt{1-x(t)} - \delta x(t))dt + \sigma(x(t))dz(t)$ with $x(0) = x_0 \in [0, 1]$ and $t \geq 0$, where $z(t)$ denotes a Wiener process.	2 (1M+1R)	FSE	infinite
He et al. [65]	$\dot{x}(t) = \rho u_1(t)\sqrt{1-x(t)} - \rho u_2(t)\sqrt{x(t)} - kx(t) + k[1 - x(t)]$ with $x(0) = x_0 \in [0, 1]$.	3 (1M+2R)	FSE	infinite
Chutani and Sethi [23]	$\dot{x}_i(t) = \rho_i u_i(t)\sqrt{1 - x_i(t) - x_j(t)} - k_i x_i(t)$, for $j = 1, 2$ and $j \neq i$.	3 (1M+2R)	FSE	infinite
Ouardighi et al. [120]	$\dot{S}(t) = \rho u(t)[(\alpha - \beta p(t))^{\gamma} - S(t)] - \delta(1 - Q(t))S(t)$, where $\gamma \in (0, 1)$, $\delta \in (0, 1)$, and $\dot{Q}(t) = \vartheta(t)[1 - Q(t)]$.	2 (1M+1R)	OLSE	finite

Table E: A summary of the after-1995 V-W extensions for the dynamic advertising problems. For our explanations on the sequence of summarizing the references and all abbreviations, see the caption of Table C.

<table>
<tr><th>Review Paper</th><th>System Dynamics</th><th>Representative References</th></tr>
<tr><td rowspan="6">Feichtinger, Hartl, and Sethi [44]</td><td>$\dot{x}_i(t) = \beta_i u_i(t)[1 - x_i(t)] - \beta_j u_j x_i(t)$: the basic Lanchester model as given in (9).</td><td>Kimball [82]</td></tr>
<tr><td>$\dot{x}_i(t) = u_i(t)[1 - x_i(t)] - \beta u_j(t) x_i(t)$ with $\beta > 0$, for $i, j = 1, 2$ and $i \neq j$.</td><td>Case [17]</td></tr>
<tr><td>$\dot{x}_i(t) = \beta_i [u_i(t)]^{\alpha_i} [1 - x_i(t)] - (\sum_{j \neq i} \beta_j [u_j(t)]^{\alpha_j}) x_i(t)$ with $\sum_{j=1}^{n} x_j = 1$, where $0 < \alpha_i < 1$, for $i = 1, \ldots, n$.</td><td>Little [91]</td></tr>
<tr><td>$\dot{S}_i(t) = f_{ij}(u_i(t)) S_j(t) - f_{ji}(u_j(t)) S_i(t) + f_i(u_i(t))$, for $i, j = 1, 2$ and $i \neq j$, where $f'_{ij}(\cdot), f'_{ji}(\cdot), f'_i(\cdot) > 0$ and $f''_{ij}(\cdot), f''_{ji}(\cdot), f''_i(\cdot) < 0$.</td><td>Erickson [34]</td></tr>
<tr><td>$x_i(t) - x_i(t-1) = f_i(u_i(t))[1 - x_i(t-1)] - f_j(u_j(t)) x_i(t-1)$, for $i, j = 1, 2$ and $i \neq j$.</td><td>Park and Hahn [123]</td></tr>
<tr><td>$\dot{x}_i(t) = \beta_i \sqrt{u_i(t)}[1 - x_i(t)] - \beta_j \sqrt{u_j(t)} x_i(t)$, for $i, j = 1, 2$ and $i \neq j$.</td><td>Chintagunta and Vilcassim [22]</td></tr>
<tr><td rowspan="2">Erickson [38]</td><td>$\dot{x}_i(t) = u_i(t)\sqrt{1 - x_i(t)} - u_j(t)\sqrt{x_i(t)}$, for $i, j = 1, 2$ and $i \neq j$.</td><td>Sorger [146]</td></tr>
<tr><td colspan="2">Other major extended Lanchester models reviewed by Erickson [38] include Case [17], Erickson [34], and Chintagunta and Vilcassim [22], which were reviewed by Feichtinger, Hartl, and Sethi [44].</td></tr>
</table>

Table F: A summary of the Lanchester advertising model and its major extensions surveyed by Erickson [38] and Feichtinger, Hartl, and Sethi [44]. Note that the review papers and the references (for each review paper) are both listed in chronological order.

References	System Dynamics	Number of Players	Solution/ Equilibrium	Time Horizon
Wang and Wu [162]	$\dot{x}_i(t) = \beta_i u_i(t)[1 - x_i(t)] - \beta_j u_j(t)x_i(t) - \gamma_i x_i(t)$ with $x_i(0) = \bar{x}_i$, for $i, j = 1, 2$ and $i \neq j$, where $x_i(t) = S_i(t)/M$.	2 (duopoly)	OLNE, CLNE	finite
Jarrar et al. [69]	$\dot{x} = \beta_i \sqrt{u_i(t)}[1 - x_i(t)] - \beta_j \sqrt{u_j(t)} x_i(t)$ with $x(0) = x_0$, for $i, j = 1, 2$ and $i \neq j$, where $\beta_i > 0$	2 (duopoly)	MNE	infinite
Breton et al. [10]	$x_{t+1} - x_t = (1 - x_t)c_l(u_{lt}) - x_t c_f(u_{ft})$, where $0 \leq x_t \leq 1$.	2 (duopoly)	FSE	infinite
Nguyen and Shi [118]	$\dot{S}_i(t) = x_i(t)\dot{S}(t) + [\beta_i \sqrt{u_i(t)} S_j(t) - \beta_j \sqrt{u_j(t)} S_i(t)]$, $x_i(t) \equiv S_i(t)/S(t)$, for $i, j = 1, 2$ and $i \neq j$, where $S(t) \equiv \sum_{k=1}^{2} S_k(t)$.	2 (duopoly)	CLNE	finite
Mesak and Calloway [104]	$\dot{S}_i(t) = -\dot{S}_j(t) = f_i(u_i(t))S_j(t) - f_j(u_j(t))S_i(t)$, for $j = 1, 2$ and $j \neq i$, where $f_i(\cdot)$ is firm i's response function such that $f_i(\cdot) > 0$, and $f_i^{'}(\cdot) > 0$; $S_1(t) + S_2(t) = M$.	2 (duopoly)	CLNE	infinite
Bass et al. [7]	$\dot{S}_i(t) = \beta_i \hat{u}_i(t)\sqrt{S_j(t)} - \beta_j \hat{u}_j(t)\sqrt{S_i(t)} + \theta_i \times \sum_{j=1}^{2}[k_j(t)u_j(t)]$, with $S_i(0) = \bar{S}_i$, for $j = 1, 2$ and $j \neq i$.	2 (duopoly)	CLNE	infinite
Bass et al. [6]	$\dot{S}_i(t) = \beta_i(t)\hat{u}_i(t)\sqrt{S_j(t)} - \beta_j(t)\hat{u}_j(t)\sqrt{S_i(t)} + \theta_i(t) \times \sum_{j=1}^{2}[k_j(t)u_j(t)]$ with $S_i(0) = \bar{S}_i$, for $j = 1, 2$ and $j \neq i$.	2 (duopoly)	CLNE	finite
Ouardighi and Pasin [121]	$\dot{x}_i(t) = \omega_i A_i(t)[1 - q_j(t)][1 - x_i(t)] - \omega_j A_j(t)[1 - q_i(t)]x_i(t)$, with $x_i(0) = \bar{x}_i \in [0, 1]$ and $\omega_i \in (0, 1)$, for $j = 1, 2$ and $j \neq i$, where $\dot{q}_i(t) = \nu_i(t)[1 - q_i(t)]$ with $q_i(t) \in (0, 1]$.	2 (duopoly)	OLNE	finite
Fruchter [52]	$\dot{S}_i(t) = \beta_i u_i(t)[M - S_i(t)] - S_i(t)\sum_{j=1, j\neq i}^{n} \beta_j u_j(t)$ with $S_i(0) = \bar{S}_i$, for $i = 1, \ldots, n$.	$n \geq 3$ (oligopoly)	CLNE	infinite
Fruchter and Kalish [56]	$\dot{x}_i(t) = \sum_{j=1}^{L}[\beta_{ij} u_{ij}(t)] - x_i(t)\sum_{k=1}^{n}\sum_{j=1}^{L}[\beta_{kj} u_{kj}(t)]$, with $x_i(0) = \bar{x}_i$, for $i = 1, \ldots, n$.	$n \geq 3$ (oligopoly)	OLNE, CLNE	infinite
Naik et al. [115]	$\dot{x}_i(t) = f_i(t) - \hat{f}(t)x_i(t)$, where $x_i(t)$ is brand i's market share at time t, $f_i(t) \equiv \alpha_i u_i(t) + \beta_i v_i(t) + \gamma_i u_i(t)v_i(t)$, and $\hat{f}(t) \equiv \sum_{j=1}^{n} f_j(t)$.	$n \geq 3$ (oligopoly)	CLNE	finite
Mesak and Darrat [107]	$\dot{S}_i(t) = f_i(u_i(t))\sum_{j=1, j\neq i}^{n} S_j(t) - \left[\sum_{j=1}^{n} f_j(u_j(t))\right] S_i(t)$, for $i = 1, \ldots, n$, where $f_i(\cdot) \geq 0$, $f_i^{'}(\cdot) \geq 0$ for $u_i(t) \geq 0$; and $\sum_{j=1}^{n} S_j(t) = M$.	$n \geq 3$ (oligopoly)	CLNE	infinite
Naik et al. [114]	$\dot{y}_i(t) = \beta_i u_i(t)\sqrt{M - y_i(t)} - \sum_{j=1, j\neq i}^{n}\left[\beta_{ij} u_j(t)\sqrt{M - y_j(t)}\right]$, for $i = 1, \ldots, n$.	$n \geq 3$ (oligopoly)	CLNE	infinite
Rubel and Zaccour [133]	$\dot{x}_i(t) = [\alpha_M u_M(t) - \alpha_R u_R(t)] - \{\phi x_i(t) - \varphi[1 - x_i(t)]\}$ with $x_i(0) = \bar{x}_i$, for $i = M, R$, where $\alpha_M, \alpha_R > 0$ and $0 \leq \phi, \varphi \leq 1$.	2 (1M+1R)	CLNE	infinite

Table G: A summary of the after-1995 Lanchester models for the dynamic advertising problems. Note that CLNE represents the closed-loop Nash equilibrium. For our explanations on the sequence of summarizing the references and other abbreviations, see the caption of Table C.

Review Paper	System Dynamics	Representative References
Sethi [138]	$\dot{S}(t) = (c+\alpha)S(t)[1 - S(t)/M(t)] - \lambda S(t)$, where $S(t)$ denotes the number of consumers who observe the information by advertisement at time t; c and α denote the number of contacts made by each consumer and the advertising coefficient, respectively; $M(t)$ is the market size at time t; and, λ is the diminishing rate.	Ozga [122]
	$S(t+\triangle t) - S(t) = u(t)[M - S(t)]\triangle t - \lambda S(t)\triangle t$, which is a reinterpretation of the V-W model in (6). Note that $S(t)$ denotes the cumulative sales till time t.	Gould [61]
Erickson [38]	$\dot{S}(t) = [\alpha_1 + \alpha_2 \ln u(t) + \alpha_3 S(t)][M - S(t)]$, where α_2 and α_3 mean the effectiveness of the advertising and the communications by the past consumers; α_1 denotes the information conveyed to new consumers through alternative means such as press reports.	Horsky and Simon [67]
	$\dot{S}_i(t) = \{\alpha_{i1} + \alpha_{i2}u_i(t) + [\alpha_{i3} + \alpha_{i4}u_i(t)]S_i(t)\}[M - S_i(t)]$, for $i = 1, \ldots, n$, where α_{i1} and α_{i3} are the coefficient of innovation and that of imitation, as defined in Bass's model in [4]; α_{i1} and α_{i3} are two parameters representing the effectiveness of advertising vis-à-vis the innovators and that of advertising toward the imitators; and, $M = \sum_{i=1}^{n} S_i(t)$.	Teng and Thompson [156]
	$\dot{S}_i(t) = [\alpha_1 + \alpha_2 \ln u_i(t) + \alpha_3 S(t)][M - S(t)]$, for $i = 1, \ldots, n$, where the parameters are defined as in [67].	Dockner and Jøgensen [30]
Feichtinger, Hartl, and Sethi [44]	$\dot{S}(t) = g(S(t), u(t))[M(p) - S(t)]$, where p denotes the retail price.	Erickson [33]
	$\dot{S}_1(t) = -u_1(t)S_1(t) - \alpha_1 S_1(t)[M - S_1(t)]/M$ with $S_1(0) = M$, $\dot{S}_3(t) = [\alpha_2 + u_2(t)][M - S_1(t) - S_3(t)] - \alpha_3 S_3(t)$ with $S_3(0) = 0$. $M = S_1(t) + S_2(t) + S_3(t)$.	Dodson and Muller [31]; Muller [112]
	$\dot{x}(t) = \{[\alpha_1 + \alpha_2 u(t)][1 - x(t)] + [\alpha_3 + \alpha_4 u(t)][1 - x(t)]x(t)\}e^{-\lambda p(t)}$, where $x(t)$ is the market share, α_i $(i = 1,2,3,4)$ is defined as α_{ij} $(j = 1,2,3,4)$ in Teng and Thompson [156], $p(t)$ is the retail price at time t, and $\lambda > 0$.	Thompson and Teng [158]
	$\dot{S}(t) = \{Mpy(t)/[u(t)S(t)/M_0] - S(t)\}k$, $\dot{y}(t) = [1 - y(t)][f(u(t)) + by(t) + \hat{b}S(t)/M_0]$.	Kalish [79]
	$\dot{S}(t) = [\alpha_1 + \alpha_2 S(t)][M(u(t), p(t)) - S(t)]$.	Mahajan and Peterson [95]
	Four diffusion models were considered: $\dot{S}(t) = f(S(t), u(t), p(t))$; $\dot{S}(t) = g(S(t), u(t))h(p(t))$; $\dot{S}(t) = [M(u(t)) - S(t)]h(p(t))$; and, $\dot{S}(t) = g(S(t), u(t))h(S(t), p(t))$.	Teng and Thompson [157]
	$\dot{S}(t) = c + \{\alpha + [\beta_0 + \beta_1 f(u(t))]S(t-1)/M\}[M - S(t-1)]$, where c and α are defined as in Ozga [122], and β_0 and β_1 are two constants.	Simon and Sebastian [145]
	$\dot{S}(t) = g(S(t), u(t))$ with $\partial g(\cdot)/\partial u(t) > 0$, $\partial^2 g(\cdot)/\partial[u(t)]^2 < 0$, and $g(\cdot) > 0$ for all $(S(t), u(t))$.	Dockner and Jøgensen [29]
	$\dot{S}(t) = [M - S(t)][\alpha_1 + \alpha_2 \ln u(t) + \alpha_3 S(t)]e^{-\lambda p(t)}$, where α_i $(i = 1,2,3)$ is defined as in Dockner and Jøgensen [30], and λ is defined as in Thompson and Teng [158].	Dockner et al. [28]

Table H: A summary of major diffusion advertising models surveyed by Erickson [38], Feichtinger, Hartl, and Sethi [44], and Sethi [138]. Note that the review papers and the references (for each review paper) are both listed in chronological order.

References	System Dynamics	Number of Players	Solution/ Equilibrium	Time Horizon
Krishnan and Jain [87]	$\dot{S}(t) = u(t)[\alpha_1 + \alpha_2 S(t)/M][M - S(t)]$.	1 (monopoly)	OS	finite
Jørgensen et al. [70]	$\dot{x}(t) = \beta u(t)[1 - x(t)] + \gamma x(t)[1 - x(t)]$, with $x(0) = 0$, $\beta > 0$, $\gamma \geq 0$, and $x(t) < 1$.	1 (monopoly)	OS	finite
Nikolopoulos and Yannacopoulos [119]	$dx(t) = \{\alpha_1[1 - x(t)] + \alpha_2 x(t)[1 - x(t)]\}dt$ $+\sigma\alpha_1[1 - x(t)]d\xi(t)$.	1 (monopoly)	OS	finite
Mesak ([101] and [106])	$\dot{S}(t) = f(S(t), p(t), u(t), r(t), t)$ with $S(0) = S_0$.	1 (monopoly)	OS	finite
Kamrad et al. [80]	$dS_i(t) = f_i(S_1(t), S_2(t), S_3(t), t)dt+$ $g_i(S_1(t), S_2(t), S_3(t), t)d\xi_i(t)$, for $i = 1, 2, 3$.	1 (monopoly)	OS	finite

Table I: A summary of the after-1995 diffusion models for the dynamic advertising problems. For our explanations on the sequence of summarizing the references and the abbreviation "OS," see the caption of Table C.

References	System Dynamics	Number of Players	Solution/ Equilibrium	Time Horizon
Piga [126]	$\dot{p}(t) = \theta[\hat{p}(t) - p(t)]$ with $p(0) = p_0$, for $i = 1, 2$, and $i \neq j$, where $\alpha_1, \theta > 0$, and $\hat{p}(t) \equiv \alpha_1 + \alpha_2[u_1(t) + u_2(t)] - [q_1(t) + q_2(t)]$.	2 (duopoly)	FNE	infinite
Lambertini and Palestini [89]	$p_i(t) = p_i^r(t) - \beta_0 q_i(t) - \beta_1 \sum_{j=1}^{m} q_j(t)$, $p_k(t) = p_c^r(t) - \beta_0 q_k(t) - \beta_1 \sum_{l=m+1,\ldots,n \text{ and } l\neq k} q_l(t)$, $\dot{p}_i^r(t) = u_i(t) + \gamma \sum_{j=1}^{m} u_j(t) - \delta p_i^r(t)$, $\dot{p}_c^r(t) = u_c(t) - \delta p_c^r(t)$, $0 < \beta_1 \leq \beta_0$, for $i = 1, \ldots, m, j = 1, \ldots, m$ and $j \neq i$.	$n \geq 3$ (oligopoly)	OLNE	infinite
Loginova [92]	$\dot{S}_0(t) = -(\alpha_1 + \alpha_2)S_0(t)$, $\dot{S}_1(t) = \alpha_1 S_0(t)$, $\dot{S}_2(t) = \alpha_2 S_0(t)$, and $S_0(t) = 1 - S_1(t) - S_2(t)$.	2 (duopoly)	NE	infinite
Loginova [92]	$\dot{S}_0(t) = -(\alpha_1 + \alpha_2)S_0(t)$, $\dot{S}_1(t) = \alpha_1 [S_0(t) + S_2(t)] - \alpha_2 S_1(t)$, $\dot{S}_2(t) = \alpha_2[S_0(t) + S_1(t)] - \alpha_1 S_2(t)$, and $S_0(t) = 1 - S_1(t) - S_2(t)$.	2 (duopoly)	NE	infinite
Fruchter and Messinger [57]	$\dot{S}(t) = g(S(t), p_d(t), p_f(t), u_d(t), u_f(t))$ with $S(0) = S_0$, $\hat{S}(t) = D(p_d(t), p_f(t), u_d(t), u_f(t)) - S(t)$ with $\partial D(\cdot)/\partial p_d(t)$, $\partial D(\cdot)/\partial p_f(t) < 0$ and $\partial D(\cdot)/\partial u_d(t)$, $\partial D(\cdot)/\partial u_f(t) \geq 0$.	2 (duopoly)	FSE	infinite

Table J: A summary of the after-1995 dynamic advertising-competition models with other attributes. For our explanations on the sequence of summarizing the references and all abbreviations, see the caption of Table C.