

The Retail Space-Exchange Problem with Pricing and Space Allocation Decisions¹

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Abstract

We consider retail space-exchange problems where two retailers exchange shelf space to increase accessibility to more of their consumers in more locations without opening new stores. Using the Hotelling model, we find two retailers' optimal prices given their host and guest space in two stores under the space-exchange strategy. Next, using the optimal space-dependent prices, we analyze a non-cooperative game where each retailer makes a space allocation decision for the retailer's own store. We show that the two retailers will implement such a strategy in the game, if and only if their stores are large enough to serve more than one-half of their consumers. Nash equilibrium for the game exists and its value depends on consumers' utilities and trip costs as well as the total available space in each retailer's store. Moreover, as a result of the space-exchange strategy, each retailer's prices in two stores are both higher than the retailer's price before the space exchange but they may or may not be identical.

Key words: Retail space-exchange, price, space allocation, Hotelling model, game theory.

1 Introduction

Can retailers selling different products implement partnership strategies that involve exchanging shelf space to improve their operating performance? Consider the following problem (and its solution) that was experienced by the British supermarket chain of food-related products known as *Waitrose* [15]: Even though its sales were increasing, many of Waitrose’s potential customers were having difficulty accessing its stores because no new stores were being opened. In order to increase accessibility to more of its customers in more locations without opening new stores, Waitrose established new channels and implemented a new business model. This was achieved by initiating strategic relationships with the British retailer of healthcare products known as *Boots*. Waitrose and Boots now stock “selective product ranges” in each others’ stores; more specifically, Waitrose’s food products are sold in Boots’s stores, while the latter retailer’s healthcare products are displayed for sale in Waitrose stores. The cooperation between Waitrose and Boots can be regarded as an implementation of the retail space-exchange strategy; see Stych [14] for a magazine article describing this partnership.

Waitrose and Boots have successfully implemented the space-exchange strategy, as indicated in a report by The Waitrose Press Center [15]. As another successful example of this novel strategy, Canada’s favourite doughnut store known as *Tim Hortons* has been working with the U.S.-based Cold Stone Creamery (a chain stores of ice cream) to implement the space-exchange strategy and operate their “co-branded” stores. This practice involves 100 stores in the U.S. and six in Canada. For more information, see Draper [5] for an article describing the partnership between Tim Hortons and Cold Stone Creamery.

With the novel retailing practice described above, when two retailers (say, 1 and 2) implement their space-exchange strategy, a retailer’s consumers can buy in either the retailer’s own store or the other retailer’s store, which means that such a strategy can result in an increased store choice for consumers. It is reasonable to expect that each consumer would buy in a store that is closer to his or her residence location, thereby incurring lower travel costs and increasing the willingness to buy at a higher retail price. Specifically, when two retailers do not exchange their space, consumers who intend to buy a retailer’s product will have to visit the retailer’s own store; but, after the space exchange, some consumers may decide to shop at the other retailer’s store (because it is closer to those consumers). This implies that the space-exchange strategy will reduce the travelling costs of some consumers who buy a retailer’s product at the

other retailer's store. Since each consumer does not need to incur a high travelling cost, he or she should be willing to buy even if the retail price is increased. Thus, both retailers 1 and 2 may respond by raising their prices to increase their profit margins. This can be regarded as the most important benefit derived from the space-exchange strategy. We find that each retailer's pricing and space allocation decisions are important to the success of the space-exchange strategy, which are the focused research questions in our paper.

One may question why the two retailers do not sell both products 1 and 2 by themselves in their own stores but instead exchange shelf space for the sale of these products. We present three reasons for this: First, when the two retailers sell identical products at their sites, they have to compete for consumers, which may result in the reduction of the two retailers' profits. Second, the two retailers have more information about their own products and are thus specialized in their product sales. If each retailer sells his own product and also the other retailer's product, then the retailer has to allocate his efforts for the sale of the product that is unfamiliar to the retailer; this may reduce the retailer's operational profitability. Third, since the retailers should have already served their markets before the space exchange, their established reputations may affect consumers' purchasing decisions. Hence, to reduce the operational risk, each retailer should optimally allocate his space to the other retailer and thus take advantage of the other retailer's reputation to efficiently provide more choices to consumers.

As the above discussion indicates, the space-exchange strategy should generate benefits to both retailers and their consumers; and, in practice, some retailers (e.g., Waitrose and Boots, Tim Hortons and Cold Stone Creamery) have already successfully implemented this strategy. Despite the apparent importance of this strategy, our literature search did not reveal any research papers dealing with space-exchange problems. There are a number of space-related publications, which do not consider the space exchange-related issues. In one publication that is closely related to our paper, Jerath and Zhang [9] consider a store-within-a-store arrangement in which a retailer rents out her retail space to two manufacturers who then have complete autonomy over retail decisions such as pricing and in-store service. The authors develop an analytical model to investigate the retailer's trade-off between channel efficiency and interbrand competition. They show that the retailer cannot credibly commit to the retail prices and service levels that two manufacturers can achieve in an integrated channel, and she should thus allow the manufacturers to set up stores within her store. In another related publication, [4], Martínez-de-Albéniz and Roels analyze a shelf-space competition problem where a single retailer optimizes

her shelf space allocation among multiple suppliers' different products based on their sales level and profit margins. The authors examine the equilibrium situation in the supply chain, and found that, in general, the retailer's and the suppliers' incentives are misaligned, resulting in suboptimal retail prices and shelf space allocations. Other recent representative space-related publications include Baron, Berman, and Perry [1], Campo, et al. [2], Kurtuluş and Toktay [10], and Wang and Gerchak [17].

In this paper, we use the Hotelling model [8] in Section 2 to analyze the space-exchange problem where retailers 1 and 2 are located at two end points of a linear city. For detailed discussions on the Hotelling model and its extensions, see, e.g., Martin [11] and Tirole [16]. The Hotelling model has been widely used to analyze marketing- and operations management-related problems. The recent representative publications, where the Hotelling models are considered, include, e.g., Dasci and Laporte [3], Ghosh and Balachander [6], Granot, Granot, and Raviv [7], and Sajeesh and Raju [12].

In our space-exchange problem, retailer i ($i = 1, 2$) sells product i to his consumers who are uniformly distributed between the two retailers. Since the success of the space-exchange strategy naturally depends on whether or not each retailer benefits from this strategy, we begin our analysis by finding each retailer's optimal pricing decision and maximum profit when they do not exchange shelf space, which are later compared with two retailers' profits under the space-exchange strategy. Next, when the two retailers decide to exchange shelf space, we first temporarily assume that, in each store, two retailers have sufficiently large space to serve all of their consumers, and calculate their corresponding optimal prices with no space (capacity) constraint.

We then find in Section 3 each retailer's optimal prices under the space constraint, i.e., the retailer's host space in his own store and his guest space allocated by the other retailer are arbitrarily given. Using two retailers' optimal space-dependent prices, we next analyze in Section 4 a non-cooperative game where each retailer maximizes his total profit in two stores to determine optimal space allocation decision for his own store and find the corresponding optimal prices for his product in two stores. We perform our best-response analysis for two retailers, and find that Nash equilibrium for the game may or may not uniquely exist, which depends on consumers' consumption utilities, trip cost, and the total space in each store. We show that, adopting the Nash equilibrium, each retailer can achieve a higher profit than before the space exchange. In Section 5, we discuss possible changes of our major results in the presence of a

“common” consumer who buys both products 1 and 2, or those when a retailer’s fixed cost of opening and staffing a new store is considered.

2 Preliminaries

As indicated by the practice of Waitrose and Boots and also by that of Tim Hortons and Cold Stone Creamery, the retail space-exchange strategy applies only when the cooperating retailers’ products are neither substitutable nor complementary, e.g., Waitrose’s food vs. Boots’s healthcare products; and, Tim Hortons’s doughnuts vs. Cold Stone Creamery’s ice cream. Thus, we can reasonably assume that the products in categories $i = 1, 2$ sold by retailer $i = 1, 2$, are neither substitutable nor complementary.

The total shelf space that is owned by retailer i is denoted by $S_i > 0$ for $i = 1, 2$. To implement the space-exchange strategy, retailer i —who sells product i —decides to allocate the retail space $S_{ij} \in [0, S_i]$ to retailer j ($j = 1, 2$ and $j \neq i$) who can then sell product j using the space S_{ij} at the site of retailer i as the “guest retailer.” As a result of the space exchange, retailer i sells product i in the remaining space $S_{ii} \equiv S_i - S_{ij}$ at his own store as the “host retailer.” As discussed in Section 1, when two retailers exchange shelf space, their customers may incur lower travel costs, and the two retailers may thus increase their retail prices without losing customers. This may be regarded as an important reason why retailers (e.g., Waitrose and Boots, Tim Hortons and Cold Stone Creamery) exchange shelf space. Accordingly, we consider the Hotelling model [8] to analyze our space-exchange problem, assuming that two retailers are located at the end points of a “linear city” of length 1, and all consumers are uniformly distributed along the city.

Since the two retailers are willing to exchange shelf space when they can enjoy more profits from the strategy, we need to compare the two retailers’ profits before and after the exchange of shelf space. We next begin by computing two retailers’ optimal prices and corresponding maximum profits when they do not exchange shelf space but only operate in their own stores.

2.1 Optimal Pricing Decision with No Space Exchange

When retailers 1 and 2 do not exchange shelf space, they sell products 1 and 2 at the retail prices p_1 and p_2 , respectively. Total number of consumers for product i is B_i , for $i = 1, 2$. In

this paper, we assume that there is no “common” consumer who intends to buy both products 1 and 2; that is, B_1 and B_2 are disjoint. In Section 5.1, we will discuss the impacts of relaxing such an assumption on our major results. As in the Hotelling model [8], each consumer incurs the transportation cost t per unit of trip length, which includes the consumer’s value of time. Let $x \in [0, 1]$ denote a point in the linear city. Assuming that the locations of retailers 1 and 2 are $x = 0$ and $x = 1$, respectively, we can calculate the trip cost of the product 1 consumer (who is served by retailer 1) at the point $x \in [0, 1]$ as tx , and also compute that of the product 2 consumer (who is served by retailer 2) at the point $x \in [0, 1]$ as $t(1 - x)$; see Figure 1. In addition, each product i ($i = 1, 2$) consumer is assumed to draw a gross utility \bar{u}_i from buying a unit of product i .

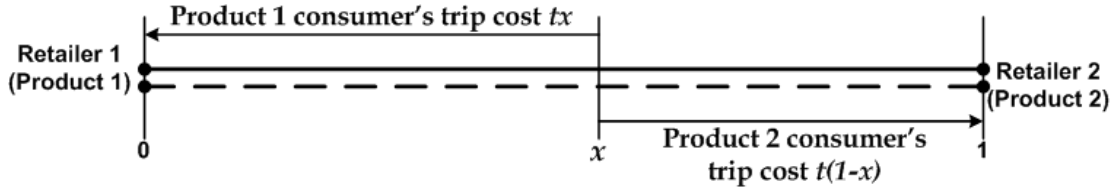


Figure 1: The trip cost of product 1 consumers (who are served by retailer 1) and that of product 2 consumers (who are served by retailer 2). Note that the solid and dashed lines between the two retailers represent the uniform distribution of product 1 consumers and that of product 2 consumers, respectively.

Using the above, we find that the product 1 consumer at the point $x \in [0, 1]$ obtains the utility \bar{u}_1 but incurs the purchase cost p_1 and the trip cost tx , and the product 2 consumer at the point $x \in [0, 1]$ gets the utility \bar{u}_2 but incurs the purchase cost p_2 and the trip cost $t(1 - x)$. It thus follows that the net utility function of the consumer at location x is calculated as,

$$u_x = \begin{cases} u_{x1} \equiv \bar{u}_1 - p_1 - tx, & \text{if product 1 bought in retailer 1's store,} \\ u_{x2} \equiv \bar{u}_2 - p_2 - t(1 - x), & \text{if product 2 bought in retailer 2's store.} \end{cases} \quad (1)$$

A product 1 consumer should be willing to buy from retailer 1 if $u_{x1} \geq 0$, or, $x \leq x_1 \equiv (\bar{u}_1 - p_1)/t$. This means that only the product 1 consumers who are located between 0 (retailer 1’s location) and x_1 should decide to buy. Naturally, retailer 1 should set his retail price p_1 such that $0 \leq x_1 \leq 1$, or $\bar{u}_1 - t \leq p_1 \leq \bar{u}_1$. Similarly, retailer 2 should determine her price p_2 such that $\bar{u}_2 - t \leq p_2 \leq \bar{u}_2$.

Then, we can calculate the demand faced by retailer i ($i = 1, 2$) as $D_i = B_i(\bar{u}_i - p_i)/t$. However, each retailer may or may not satisfy his demand, because he only has the space S_i to

stock product i . Assuming that each retailer can display one unit of his product on a unit of the retail space, we find that retailer i can realize the sales $\min[B_i(\bar{u}_i - p_i)/t, S_i]$, and achieve the profit as,

$$\pi_i = (p_i - c_i) \min[B_i(\bar{u}_i - p_i)/t, S_i] = B_i(p_i - c_i) \min[(\bar{u}_i - p_i)/t, S_i/B_i], \quad (2)$$

where c_i denotes retailer i 's unit acquisition cost. To determine the optimal price p_i^* for retailer i , we must solve the constrained maximization problem, $\max_{\bar{u}_i - t \leq p_i \leq \bar{u}_i} \pi_i$.

Lemma 1 When two retailers do not exchange shelf space, the optimal prices and maximum profits for retailer i ($i = 1, 2$) can be found as follows:

Conditions		Optimal Price	Maximum Profit
$S_i \geq B_i$	$c_i + 2t \geq \bar{u}_i$	$(\bar{u}_i + c_i)/2$	$B_i(\bar{u}_i - c_i)^2/(4t)$
	$c_i + 2t \leq \bar{u}_i$	$\bar{u}_i - t$	$B_i(\bar{u}_i - t - c_i)$
$S_i < B_i$	$c_i + 2tS_i/B_i \geq \bar{u}_i$	$(\bar{u}_i + c_i)/2$	$B_i(\bar{u}_i - c_i)^2/(4t)$
	$c_i + 2tS_i/B_i \leq \bar{u}_i$	$\bar{u}_i - tS_i/B_i$	$[\bar{u}_i - tS_i/B_i - c_i]S_i$

Proof. For a proof of this lemma and the proofs of all subsequent lemmas, see online Appendix A. ■

2.2 Optimal Retail Prices under the Space-Exchange Strategy with Sufficiently-Large Host and Guest Spaces

We consider the two retailers' optimal pricing decisions when they decide to exchange shelf space. Now, we *temporarily* assume that each retailer's host space and guest space are large enough to serve all of the retailer's consumers; and, under this assumption, we compute the retailer's optimal prices. After the space exchange, each consumer can buy in either retailer 1's store or retailer 2's store, which depends on from which store the consumer can draw a higher net utility. Consider the product i ($i = 1, 2$) consumer who resides at the point $x \in [0, 1]$ and decides to buy a unit of product i from retailer i at either his host space [in retailer i 's store] or his guest space [in retailer j 's ($j = 1, 2$ and $j \neq i$) store]. We compute the consumer's utilities

drawn from purchasing from two stores as,

$$\hat{u}_{xi} = \begin{cases} \hat{u}_{xi1} \equiv \bar{u}_i - p_{i1} - tx, & \text{if product } i \text{ bought in retailer 1's store,} \\ \hat{u}_{xi2} \equiv \bar{u}_i - p_{i2} - t(1-x), & \text{if product } i \text{ bought in retailer 2's store,} \end{cases} \quad (3)$$

where \hat{u}_{xij} denotes the product i consumer's net utility drawn from buying at retailer j 's store, and p_{ij} represents the retail price of product i in retailer j 's store.

Similar to Section 2.1, we can compute the demands faced by retailer i in two stores, as given in the following remark. For a detailed discussion, see online Appendix C.

Remark 1 We find the demands for retailer i 's product ($i = 1, 2$) as follows:

1. If $p_{i1} + p_{i2} \leq 2\bar{u}_i - t$, then the demands faced by retailer i in retailer 1's and retailer 2's stores are computed as $D_{i1} = B_i(p_{i2} - p_{i1} + t)/(2t)$ and $D_{i2} = B_i(p_{i1} - p_{i2} + t)/(2t)$, respectively. Note that $D_{i1} + D_{i2} = B_i$.
2. If $p_{i1} + p_{i2} > 2\bar{u}_i - t$, then the demands faced by retailer i are computed as $D_{i1} = B_i(\bar{u}_i - p_{i1})/t$ and $D_{i2} = B_i(\bar{u}_i - p_{i2})/t$. The total demand for product i is thus $D_{i1} + D_{i2} = B_i(2\bar{u}_i - p_{i1} - p_{i2})/t \leq B_i$.

From the above we find that all product i consumers will buy when $p_{i1} + p_{i2} \leq 2\bar{u}_i - t$ whereas some consumer(s) may not buy when $p_{i1} + p_{i2} \geq 2\bar{u}_i - t$. Note that, if $p_{i1} + p_{i2} = 2\bar{u}_i - t$, then the demands for the above two cases are the same. \triangleleft

The above remark indicates that retailer i can set suitable retail prices p_{i1} and p_{i2} to affect consumers' purchasing decisions. That is, if the retailer does not have sufficient space in two retailers' stores, then he may determine his retail prices under the condition that $p_{i1} + p_{i2} \geq 2\bar{u}_i - t$. Otherwise, if the retailer's shelf space in the two stores is large enough to satisfy B_i product i consumers, then the retailer may need to consider the condition that $p_{i1} + p_{i2} \leq 2\bar{u}_i - t$ to make his pricing decisions.

Lemma 2 When retailer i ($i = 1, 2$) has sufficiently large host and guest shelf space, we find his optimal prices (p'_{i1}, p'_{i2}) and resulting demands (D_{i1}, D_{i2}) in two stores, and compute his maximum profit, as given in Table 1. \blacksquare

As the above lemma indicates, retailer i may need to determine his prices in two stores to serve some, rather than all, of B_i product i consumers, if the product i consumer residing at

Condition	Optimal Prices	Demands	Maximum Profit
$\bar{u}_i > c_i + t$	$p'_{i1} = p'_{i2} = \bar{u}_i - \frac{t}{2}$	$D_{i1} = D_{i2} = \frac{B_i}{2}$	$B_i \left(\bar{u}_i - c_i - \frac{t}{2} \right)$
$\bar{u}_i \leq c_i + t$	$p'_{i1} = p'_{i2} = \frac{\bar{u}_i + c_i}{2}$	$D_{i1} = D_{i2} = B_i \frac{\bar{u}_i - c_i}{2t}$	$B_i \frac{(\bar{u}_i - c_i)^2}{2t}$

Table 1: Retailer i 's optimal prices p'_{i1} and p'_{i2} , and the resulting demands D_{i1} and D_{i2} in retailer 1's and retailer 2's stores, respectively; and the retailer's maximum profit generated in the two stores.

the location of retailer j 's store ($j = 1, 2, j \neq i$) cannot enjoy a positive net utility from buying product i in retailer i 's store, i.e., $\bar{u}_i \leq c_i + t$.

3 Optimal Prices Given the Space Allocation Decisions

In this section, we consider two retailers' optimal pricing decisions given the space-allocation decisions in two stores. (The results here differ from those in Section 2.2 where we determine the two retailers' optimal prices assuming that they have sufficient host and guest space.) Subsequently, using each retailer's optimal *space-dependent* prices in two stores, we find the optimal allocation of the total space S_i ($i = 1, 2$) between the two retailers.

Next, we determine retailer i 's ($i = 1, 2$) optimal pricing decisions (p_{i1} and p_{i2}) given the host space S_{ii} in his own store and the guest space $S_{ji} = S_j - S_{jj}$ in retailer j 's ($j = 1, 2, j \neq i$) store. Since one unit of product i is carried per unit of shelf space, the total number of product i available for sale in two stores can be calculated as $T_i \equiv S_{1i} + S_{2i}$. Note that each retailer's maximum available products in each store can be regarded as the "capacity" for the retailer, who should thus make his or her optimal pricing decisions under the capacity constraint.

As Lemma 2 indicates, retailer i 's optimal pricing decisions with no capacity constraint depend on the comparison between \bar{u}_i and $c_i + t$. Accordingly, we consider the two cases, (i) $\bar{u}_i > c_i + t$, and (ii) $\bar{u}_i \leq c_i + t$; and for each case, we find retailer i 's optimal prices under the capacity constraints (i.e., at most S_{ii} and S_{ji} units of product i are available for sale in retailer i 's own store and in retailer j 's store).

3.1 Optimal Prices when $\bar{u}_i > c_i + t$

If $\bar{u}_i > c_i + t$, then we learn from Lemma 2 that retailer i should make his pricing decisions to serve all B_i product i consumers, which requires this retailer to have a sufficiently-large

space to stock B_i units of product i . Additionally, since retailer i 's profit is maximized when $S_{ii} = S_{ji} = B_i/2$, his *desired* host space and guest space should be both equal to $B_i/2$. However, the space allocated to retailer i in each store may be different from $B_i/2$, and the total space for retailer i in two stores may or may not be large enough to serve all of B_i consumers. More specifically, if the total space for retailer i is given such that $S_{ii} + S_{ji} \geq B_i$, then the total demand B_i can be satisfied; but, if $S_{ii} + S_{ji} < B_i$, then only a part of the demand will be fulfilled.

Lemma 3 Suppose that retailer i 's ($i = 1, 2$) host space and guest space are given as $S_{ii} \in [0, S_i]$ and $S_{ji} \in [0, S_j]$, for $j = 1, 2$, and $j \neq i$. If $\bar{u}_i > c_i + t$, then retailer i 's optimal prices in two stores are found as follows:

1. If retailer i 's total space $S_{ii} + S_{ji}$ in two stores is large enough to serve all of B_i product i consumers, i.e., $S_{ii} + S_{ji} \geq B_i$, then the retailer's optimal prices in his own store and retailer j 's store—denoted by p_{ii}^* and p_{ij}^* , respectively—are determined as given in Table 2.

Conditions	p_{ii}^*	p_{ij}^*
$S_{ii} \geq B_i/2$ $S_{ji} \geq B_i/2$	$\bar{u}_i - \frac{t}{2}$	$\bar{u}_i - \frac{t}{2}$
$S_{ii} < B_i/2$ $S_{ji} > B_i/2$	$\bar{u}_i - t \frac{S_{ii}}{B_i}$	$\max \left[\frac{\bar{u}_i + c_i}{2}, \bar{u}_i - t \left(1 - \frac{S_{ii}}{B_i} \right) \right]$
$S_{ii} > B_i/2$ $S_{ji} < B_i/2$	$\max \left[\frac{\bar{u}_i + c_i}{2}, \bar{u}_i - t \left(1 - \frac{S_{ji}}{B_i} \right) \right]$	$\bar{u}_i - t \frac{S_{ji}}{B_i}$

Table 2: Retailer i 's optimal price p_{ii}^* and p_{ij}^* for product i in his host space $S_{ii} \in [0, S_i]$ and his guest space $S_{ji} \in [0, S_j]$, respectively, when $\bar{u}_i > c_i + t$ and $S_{ii} + S_{ji} \geq B_i$.

2. If $S_{ii} + S_{ji} < B_i$, then the retailer's optimal pricing decisions are obtained as, $p_{ii}^* = \max[(\bar{u}_i + c_i)/2, \bar{u}_i - tS_{ii}/B_i]$ and $p_{ij}^* = \max[(\bar{u}_i + c_i)/2, \bar{u}_i - tS_{ji}/B_i]$. ■

Next, we compare retailer i 's optimal price with no space exchange—as given in Lemma 1—and the optimal price under the space-exchange strategy—as given in Lemma 3, in order to examine the impact of the strategy on the retailer's pricing decision.

Lemma 4 When $\bar{u}_i > c_i + t$ ($i = 1, 2$) and retailer i 's host space and guest space are given as $S_{ii} \in [0, S_i]$ and $S_{ji} \in [0, S_j]$ ($j = 1, 2, j \neq i$), respectively, the retailer's optimal price p_{ii}^* in his own store is greater than his optimal price p_i^* that is obtained when there is no space exchange,

i.e., $p_{ii}^* > p_i^*$. However, retailer i 's optimal price p_{ij}^* in retailer j 's store may or may not be greater than p_i^* . Specifically, if the total space S_i in retailer i 's own store is larger than or equal to $B_i(\bar{u}_i - c_i)/(2t)$, i.e., $S_i \geq B_i(\bar{u}_i - c_i)/(2t)$, then p_{ij}^* is always greater than p_i^* . Otherwise, then p_{ij}^* may not be greater than p_i^* . ■

This lemma says that, if retailer i has a sufficiently large space in his own store, i.e., $S_i \geq B_i(\bar{u}_i - c_i)/(2t)$, then his prices in two stores under the space-exchange strategy should be higher than the price when two retailers do not exchange shelf space. Moreover, we note that $B_i(\bar{u}_i - c_i)/(2t) > B_i/2$ because $\bar{u}_i > c_i + t$. This means that the prices should rise as a result of the space-exchange strategy, if the total available space S_i in retailer i 's store is large enough to satisfy more than a half of product i consumers (including some consumers who are closer to retailer j 's store). This interesting result may be justified as follows: After the space exchange, those consumers closer to retailer j 's store could visit retailer j 's store to buy product i . That is, retailer i may serve fewer consumers in his own store, and may thus raise his retail price to increase his profit.

Remark 2 Lemma 3 was used for comparing retailer i 's ($i = 1, 2$) prices under different conditions. We can use the same lemma to calculate the retailer's maximum profit as given in Table 3. ◁

Conditions		Retailer i 's Maximum Profit
$S_{ii} + S_{ji} \geq B_i$	$S_{ii} \geq B_i/2, S_{ji} \geq B_i/2$	$\pi_i^1 \equiv B_i(\bar{u}_i - c_i - t/2)$
	$S_{ii} < \frac{B_i}{2}$ $S_{ji} > \frac{B_i}{2}$	$\pi_i^2 \equiv B_i(\bar{u}_i - c_i - t) + 2tS_{ii}(1 - S_{ii}/B_i)$
	$S_{ii} > \frac{B_i}{2}$ $S_{ji} < \frac{B_i}{2}$	$\pi_i^4 \equiv B_i(\bar{u}_i - c_i - t) + 2tS_{ji}(1 - S_{ji}/B_i)$
	$S_{ii} \leq B_i \left(1 - \frac{\bar{u}_i - c_i}{2t}\right)$ $S_{ii} \leq B_i \left(1 - \frac{\bar{u}_i - c_i}{2t}\right)$	$\pi_i^3 \equiv B_i(\bar{u}_i - c_i)^2/(4t) + (\bar{u}_i - tS_{ii}/B_i - c_i)S_{ii}$ $\pi_i^5 \equiv B_i(\bar{u}_i - c_i)^2/(4t) + (\bar{u}_i - tS_{ji}/B_i - c_i)S_{ji}$
$S_{ii} + S_{ji} < B_i$	$S_{ii} \leq B_i \frac{\bar{u}_i - c_i}{2t}$ $S_{ii} \geq B_i \frac{\bar{u}_i - c_i}{2t}$	$\pi_i^6 \equiv (\bar{u}_i - tS_{ii}/B_i - c_i)S_{ii} + (\bar{u}_i - tS_{ji}/B_i - c_i)S_{ji}$ $\pi_i^7 \equiv B_i(\bar{u}_i - c_i)^2/(4t) + (\bar{u}_i - tS_{ji}/B_i - c_i)S_{ji}$
	$S_{ii} \geq B_i \frac{\bar{u}_i - c_i}{2t}$ $S_{ii} \leq B_i \frac{\bar{u}_i - c_i}{2t}$	$\pi_i^8 \equiv B_i(\bar{u}_i - c_i)^2/(4t) + (\bar{u}_i - tS_{ii}/B_i - c_i)S_{ii}$
	$S_{ii} \leq B_i \frac{\bar{u}_i - c_i}{2t}$ $S_{ii} \geq B_i \frac{\bar{u}_i - c_i}{2t}$	

Table 3: Retailer i 's maximum profit when $\bar{u}_i > c_i + t$.

In Section 2.1 we found retailer i 's optimal pricing decision and computed the corresponding maximum profit when the two retailers do not exchange shelf space. We now compare retailer i 's maximum profit (i) with no space exchange and (ii) with space exchange strategy, in order to examine whether or not two retailers can benefit from the strategy.

Corollary 1 If the two retailers implement the space-exchange strategy when $\bar{u}_i > c_i + t$, then we find that π_i^i ($i = 1, \dots, 8, i \neq 6$) in Table 3 is greater than retailer i 's profit when two retailers do not exchange shelf space. However, π_i^6 in Table 3 may be smaller than the retailer's profit with no space exchange. ◀

As the above corollary indicates, retailer i ($i = 1, 2$) may be worse off under the space-exchange strategy, if he cannot use his host space and guest space to serve all of B_i product i consumers (i.e., $S_{ii} + S_{ji} < B_i$), and both the host space and the guest space are smaller than a threshold value [i.e., $S_{ii}, S_{ji} \leq B_i(\bar{u}_i - c_i)/(2t)$]. This means that, in order to cooperate for such a strategy, retailer i should retain a sufficiently large host space and retailer j ($j = 1, 2, j \neq i$) must also allocate a sufficiently large guest space to retailer i . However, even though π_i^6 in Table 3 may be smaller than the profit with no space exchange, retailer i should still be better off from implementing the space-exchange strategy because he can choose to allocate S_{ii} units to himself. For example, the retailer can increase S_{11} to a level such that his profit is π_i^7 , which is higher than the retailer's profit when there is no space exchange.

3.2 Optimal Prices when $\bar{u}_i \leq c_i + t$

When $\bar{u}_i \leq c_i + t$ ($i = 1, 2$), we found in Lemma 2 that, if there is no capacity constraint, retailer i 's profit is maximized when $p_{ii} = p_{ij} = (\bar{u}_i + c_i)/2$ ($j = 1, 2, j \neq i$) and the corresponding demands in two stores are $D_{ii} = D_{ij} = B_i(\bar{u}_i - c_i)/(2t)$. We now investigate the retailer's optimal pricing decisions given his host space and guest space.

Lemma 5 Suppose that retailer i 's ($i = 1, 2$) host space and guest space are given as $S_{ii} \in [0, S_i]$ and $S_{ji} \in [0, S_j]$, for $j = 1, 2, j \neq i$. If $\bar{u}_i \leq c_i + t$, retailer i 's optimal prices in two stores are found as $p_{ii}^* = \max[(\bar{u}_i + c_i)/2, \bar{u}_i - tS_{ii}/B_i]$ and $p_{ij}^* = \max[(\bar{u}_i + c_i)/2, \bar{u}_i - tS_{ji}/B_i]$. If $S_i \geq B_i(\bar{u}_i - t)/(2t)$, then retailer i 's optimal prices p_{ii}^* and p_{ij}^* under the space-exchange strategy are both higher than his optimal price p_i^* when two retailers do not exchange shelf

space. However, if $S_i < B_i(\bar{u}_i - c_i)/(2t)$, then p_{ii}^* is always greater than p_i^* ; but, p_{ij}^* may or may not be greater than p_i^* . We find that $p_{ij}^* > p_i^*$ when $S_{ji} \leq S_i$. ■

Lemma 5 gives us retailer i 's pricing decisions in two stores, when the product i consumer residing at the location of a retailer's store cannot enjoy a positive net utility if he or she decides to buy in the other retailer's store (i.e., $\bar{u}_i \leq c_i + t$). As Lemma 5 implies, the retailer should increase his price in his own store, when he cooperates with the other retailer for the space-exchange strategy. However, after retailer i also operates in retailer j 's store using the guest space S_{ji} , his price for product i in retailer j 's store may be lower than that in retailer i 's own store before the space exchange. More specifically, if the total space S_i in retailer i 's store is sufficiently large [i.e., $S_i \geq B_i(\bar{u}_i - t)/(2t)$], then, no matter what the guest space S_{ji} —that is allocated by retailer j to retailer i —is, retailer i should always set the price p_{ij} higher than p_i^* . Otherwise, if retailer i cannot use the total space S_i in his own store to serve a half of B_i product i consumers, i.e., $S_i < B_i(\bar{u}_i - c_i)/(2t)$, then the retailer may or may not set a price higher than p_i^* , which depends on the value of the guest space S_{ji} . If the guest space is larger than the total space S_i in retailer i 's own store, then retailer i may choose a price lower than p_i^* in order to entice more consumers to buy product i in retailer j 's store, because the space in the retailer's own store is very small. Otherwise, if the guest space S_{ji} is also very small (i.e., $S_{ji} < S_1$), then retailer i is unable to serve all (or even, most of) product i consumers, and should thus increase the prices in two stores to improve his profit.

Lemmas 4 and 5 indicates the comparison between retailer i 's optimal prices with and without the space exchange, for the case that $\bar{u}_i > c_i + t$ and the case that $\bar{u}_i \leq c_i + t$, respectively. Using these results, we reach a conclusion regarding the impacts of the space-exchange strategy on the retail prices, as given in the following proposition.

Proposition 1 When retailers 1 and 2 implement the space-exchange strategy, each retailer's prices for his products in two stores are higher than the retailer's price in his own store when two retailers do not exchange shelf space.

Proof. A proof for this proposition and our proofs for all subsequent propositions are provided in online Appendix B. ■

The above proposition indicates that, if two retailers decide to exchange shelf space, then they should raise their retail prices. Using Lemmas 3 and 5, we can also find the following result regarding each retailer's two prices in two stores.

Proposition 2 Retailer i ($i = 1, 2$) may determine different prices for product i in two stores. That is, after two retailers exchange shelf space, the retail prices of the same products at two stores may not be identical. ■

We also note from Lemmas 3 and 5 that retailer i 's optimal pricing decisions when $\bar{u}_i \leq c_i + t$ are the same as those when $\bar{u}_i > c_i + t$ and $S_{ii} + S_{ji} < B_i$. Thus, the retailer's possible maximum profits when $\bar{u}_i \leq c_i + t$ should include those when $S_{ii} + S_{ji} < B_i$ in Table 3. Similar to Corollary 1, we find that, when $\bar{u}_i \leq c_i + t$, π_i^k ($k = 7, 8$)—as given in Table 3—is greater than retailer i 's profit when two retailers do not exchange shelf space; but, π_i^6 in Table 3 may be smaller than the retailer's profit with no space exchange, which depends on the values of S_{ii} and S_{ji} .

In addition to π_i^k ($k = 6, 7, 8$) in Table 3, retailer i may achieve the maximum profit $\pi_i^9 \equiv B_i(\bar{u}_i - c_i)^2/(2t)$, which occurs when $\bar{u}_i \leq c_i + 2tS_{ii}/B_i$ and $\bar{u}_1 \leq c_1 + 2tS_{21}/B_1$. Note that π_i^9 is not considered for the case that $S_{ii} + S_{ji} < B_i$ in Table 3, because the conditions that $\bar{u}_i \leq c_i + 2tS_{ii}/B_i$ and $\bar{u}_i \leq c_i + 2tS_{ji}/B_i$ imply that $S_{ii} + S_{ji} \geq B_i$, which is contrary to the case that $S_{ii} + S_{ji} < B_i$.

Similar to our previous discussion for the case that $\bar{u}_i > c_i + t$, we find that, if retailer i 's host space and guest space are both small, then the retailer may not achieve a higher profit from implementing the space-exchange strategy and may thus lose the incentive to cooperate with retailer j . On the other hand, if the retailer has a sufficiently large space in his own store and/or retailer j 's store, then he should obtain a profit that is higher than the profit with no space exchange. That is, in order to entice retailer i to exchange his space with retailer j , retailer j may need to allocate a sufficiently large space to retailer i .

4 Nash Equilibrium Space-Allocation Decisions

We now investigate the optimal allocation of each store's shelf space between two retailers in the equilibrium. That is, we determine the optimal values of S_{ii} and S_{ij} in retailer i 's store where $S_{ii} + S_{ij} = S_i$, for $i, j = 1, 2$ and $i \neq j$. Since the total space in each store (i.e., S_i) is given, retailer i only needs to determine the value of S_{ii} and allocates the remaining space $S_{ij} = S_i - S_{ii}$ to retailer j . To find the optimal space decision, each retailer should maximize the sum of his profits generated in two stores. Thus, the space allocation problem can be naturally regarded as a “simultaneous-move” non-cooperative game, and the two retailers' optimal space

allocation decisions should be characterized by Nash equilibrium.

To solve the non-cooperative game and find the Nash equilibrium, we need to first analyze each retailer's best response—i.e., the optimal space decision for a given space allocation decision of the other retailer. Next, we begin by finding retailer i 's best space allocation decision for his own store, assuming that retailer j decides to retain the space S_{jj} and allocate the space S_{ji} to retailer i .

4.1 The Best-Response Analysis

To implement the space-exchange strategy, retailer i ($i = 1, 2$) uses the guest space S_{ji} —given by retailer j ($j = 1, 2, j \neq i$)—to serve some or all product i consumers, and determines his host space $S_{ii} \in [0, S_i]$ and allocate the space $S_{ij} = S_i - S_{ii}$ to retailer j . Thus, to find the best response to retailer j 's space allocation decision, retailer i should find the optimal host space that maximizes his own profit. As discussed in Section 3, we calculate retailer i 's maximum profit given the retailer's host space and guest space, as shown in Table 3.

Next, we assume that the guest space $S_{ji} \in [0, S_j]$ is given, and use our results in Table 3 to find the optimal host space S_{ii} (best response) for retailer i . Because our analysis in Section 3 indicates that retailer i 's optimal pricing decisions depends on the comparison between \bar{u}_i and $c_i + t$, we consider the retailer's optimal space allocation decision for the two cases: $\bar{u}_i > c_i + t$ and $\bar{u}_i \leq c_i + t$.

4.1.1 The Best Response when $\bar{u}_i > c_i + t$

We now determine retailer i 's best space allocation decision, when the product i consumer residing at the site of retailer j 's store can gain a positive net utility if he or she visits retailer i 's store to buy product i (i.e., $\bar{u}_i > c_i + t$).

Lemma 6 When $\bar{u}_i > c_i + t$, retailer i 's optimal space allocation decision depends on the values of S_{ji} and the total space S_i in his own store, as shown in Table 4. ■

From the above lemma, we learn that, if either the total space S_i in retailer i 's own store or the retailer's guest space S_{ji} (allocated by retailer j) cannot be used to serve a half of B_1 product 1 consumers, then retailer i should make his space allocation decision to serve a part (rather than all) of B_i consumers. Otherwise, if both S_i and S_{ji} are large enough to serve a

Conditions		Optimal Space Decision	
$S_i + S_{ji} \geq B_i$	$S_{ji} \geq B_i/2$	$S_i \geq B_i[1 - (\bar{u}_i - c_i)/(2t)]$	$S_{ii}^* = \min[S_i, B_i/2],$ $S_{ij}^* = [0, S_i - B_i/2]^+.$ (All consumers buy.)
		$S_i < B_i[1 - (\bar{u}_i - c_i)/(2t)]$	$S_{ii}^* = S_i$ and $S_{ij}^* = 0.$ (Some consumers may not buy.)
	$B_i[1 - (\bar{u}_i - c_i)/(2t)] \leq S_{ji} < B_i/2$		$S_{ii}^* = B_i - S_{ji},$ $S_{ij}^* = S_i - B_i + S_{ji}.$ (All consumers buy.)
	$S_{ji} < B_i[1 - (\bar{u}_i - c_i)/(2t)]$		$S_{ii}^* = B_i(\bar{u}_i - c_i)/(2t),$ $S_{ij}^* = S_i - B_i(\bar{u}_i - c_i)/(2t).$ (Some consumers may not buy.)
$S_i + S_{ji} < B_i$		$S_{ii}^* = \min[B_i(\bar{u}_i - c_i)/(2t), S_i],$ $S_{ij}^* = [0, S_i - B_i(\bar{u}_i - c_i)/(2t)]^+.$ (Some consumers may not buy.)	

Table 4: Retailer i 's best-response space decision when $\bar{u}_i > c_i + t$.

half of product i consumers, then the retailer should determine his host space such that all consumers will buy in two stores. This may reflect the fact that retailer i *intends* to serve all product i consumers using his space in two stores. Thus, if retailer j allocates a sufficiently large space to retailer i and the total space S_i in retailer i 's store is also sufficiently large, then retailer i should decide to retain a host space that is large enough to assure that he can serve all consumers in two stores.

However, if retailer i cannot serve $B_i/2$ consumers in his guest space S_{ji} , then he should not retain a large host space to serve all consumers, which may be justified as follows: When the guest space is so small that less than a half of consumers are willing to buy, retailer i should have to use his host space to serve more than a half of B_i consumers if he intends to serve all consumers in two stores. But, in order to sell more than $B_i/2$ units of product i in retailer i 's own store, the retailer has to reduce his retail price to a low level, which may thus reduce his total profit. Similarly, if S_i is small, then retailer i should accept a small guest space from retailer j and should not serve all consumers.

4.1.2 The Best Response when $\bar{u}_i \leq c_i + t$

From Lemmas 3 and 5 we find that, if $\bar{u}_i \leq c_i + t$, then retailer i 's optimal prices in two stores are the same as those when $\bar{u}_i > c_i + t$ and $S_{ii} + S_{ji} < B_i$. Thus, the retailer's maximum profit when $\bar{u}_i \leq c_i + t$ is the same as that when $S_{ii} + S_{ji} < B_i$ in Table 3.

Lemma 7 If $\bar{u}_i \leq c_i + t$ ($i = 1, 2$), then retailer i 's best-response space allocation decision is the same as that when $S_i + S_{ji} < B_i$ in last row of Table 4. ■

As the above lemma indicates, retailer i ($i = 1, 2$) should make his space allocation decision to serve a part of B_i product i consumers if the product i consumer at the site of retailer j 's ($j = 1, 2, j \neq i$) store cannot enjoy a positive net utility when he or she buys in retailer i 's store, i.e., $\bar{u}_i \leq c_i + t$. For this case, if two retailers do not implement the space-exchange strategy, then some consumers who are closer to the site of retailer j do not buy product i . After the space exchange, retailer i may need to utilize his guest space to serve those consumers (who do not buy before the space exchange). Note that, in the linear city, a half of total product i consumers (i.e., B_i consumers) are closer to retailer j 's store, and they could prefer to buy in retailer j 's rather than retailer i 's store. This means that, as a consequence of the space exchange, retailer i may serve less consumers in his store and may thus raise the retail price in his own store to increase his profit margin. In addition, to assure the retailer's profit in retailer j 's store, the retailer should not reduce his price for product i in retailer j 's store to a low level, which may discourage some consumers from buying product i . Therefore, to maximize retailer i 's total profit in two stores, the retailer may make his pricing and space allocation decisions to only serve a part of B_i consumers.

4.2 Nash Equilibrium

We use our above best-response analysis for two retailers to find the Nash equilibrium (S_{ii}^N, S_{jj}^N) ($i, j = 1, 2, i \neq j$) for the non-cooperative game. Note that the guest space S_{ij}^N and S_{ji}^N in two stores are computed as $S_{ij}^N = S_i - S_{ii}^N$ and $S_{ji}^N = S_j - S_{jj}^N$; and, after each retailer makes his or her optimal space decision, the retailer's optimal price is correspondingly determined using our results in Section 3.

We find from our best-response analysis that in some cases, a retailer may allocate zero space to the other retailer. For example, if $\bar{u}_i - c_i > t$, $S_i + S_{ji} \geq B_i$, $S_{ji} \geq B_i/2$, and $B_i[1 - (\bar{u}_i - c_i)/(2t)] \leq S_i \leq B_i/2$, then retailer i should not allocate any space to retailer j , as indicated by Lemma 6. Naturally, to implement the space-exchange strategy, each retailer must allocate some nonzero space to the other retailer. Therefore, if a retailer does not allocate any space to the other retailer, then two retailers should not consider the space-exchange strategy but instead operate with no space exchange.

Proposition 3 In the Nash equilibrium, retailers 1 and 2 should decide to implement the space-exchange strategy if and only if $S_i > \max[B_i[1 - (\bar{u}_i - c_i)/(2t)], B_i(\bar{u}_i - c_i)/(2t)]$, for $i = 1, 2$. ■

In the above proposition, we note that, for retailer i ($i = 1, 2$), either $[1 - (\bar{u}_i - c_i)/(2t)]$ or $(\bar{u}_i - c_i)/(2t)$ must be greater than or equal to $1/2$. That is, for the “simultaneous-mover” game, the total shelf space S_i in retailer i ’s own store must be large enough to serve more than a half of B_i product i consumers, in order to let retailer i have an incentive for the space exchange with retailer j ($j = 1, 2, j \neq i$). Thus, two retailers should have sufficient shelf space in their own stores in order to implement the space-exchange strategy. Otherwise, they may have no incentive for the space exchange in the game.

One may note that two retailers with small shelf space could also consider the space-exchange strategy. For example, suppose that retailer i ($i = 1, 2$) can stock only two units of product i in his store before the space exchange, i.e., $S_i = 2$, for $i = 1, 2$. When two retailers do not exchange shelf space, retailer i would set his price such that the *two* consumers who are the closest to the retailer along the Hotelling line would find it worthwhile to buy product i . If two retailers exchange shelf space, then they may raise their prices without losing any consumers, and their profits could thus be higher than those in the “no space exchange” case. This differs from Proposition 3, which is justified as follows: Proposition 3 holds when two retailers make their decisions in the *non-cooperative* game whereas the above discussion is based on the assumption that two retailers jointly make their decisions in the *cooperative* setting. For a detailed discussion, see online Appendix E. Note that we use the Hotelling model to analyze the space-exchange problem; thus, Proposition 3 applies to the non-cooperative setting.

Since two retailers decide to exchange shelf space in the “simultaneous-move” game if and only if the non-zero space allocation decisions exist in Nash equilibrium, we next analyze our non-cooperative space-exchange game to find Nash equilibrium under the condition in Proposition 3. We learn from our previous analysis that two retailers make their pricing and space decisions according to whether or not all consumers for each product can enjoy a positive net utility from buying at each end point (i.e., the site of each retailer’s store) of the linear city. Accordingly, we should compare \bar{u}_i and $c_i + t$ for retailer i , in order to compute the Nash equilibrium for the space-exchange problem. Hence, for our game analysis, we need to consider the following three cases: (i) $\bar{u}_i \leq c_i + t$, for $i = 1, 2$; (ii) $\bar{u}_i > c_i + t$ and $\bar{u}_j \leq c_j + t$, for $i, j = 1, 2$,

$i \neq j$; and (iii) $\bar{u}_i > c_i + t$, for $i = 1, 2$.

Our best-response analysis indicates that, when $\bar{u}_i > c_i + t$, retailer i has a number of different optimal space decisions dependent on the space in his own store and his guest space. Therefore, for Case (iii), there should be a number of possible Nash equilibria, which depend on the total space in each retailer's store. In order to facilitate our discussion, we first consider Case (i), and find the corresponding Nash equilibrium. This is then followed by our discussion by the remaining two cases.

4.2.1 Nash Equilibrium when $\bar{u}_i \leq c_i + t$ ($i = 1, 2$)

Using our best-response analysis in Section 4.1, we now solve the two-person non-cooperative game to find the Nash equilibrium for retailers 1 and 2.

Lemma 8 If $\bar{u}_i \leq c_i + t$ ($i = 1, 2$), then Nash equilibrium *uniquely* exists as $S_{ii}^N = B_i(\bar{u}_i - c_i)/(2t)$ and $S_{jj}^N = B_j(\bar{u}_j - c_j)/(2t)$, for $j = 1, 2$ and $j \neq i$. Retailer i should allocate the space $S_{ij}^N = S_i - B_i(\bar{u}_i - c_i)/(2t) > 0$ to retailer j , who allocates the space $S_{ji}^N = S_j - B_j(\bar{u}_j - c_j)/(2t) > 0$ to retailer i .

To implement the equilibrium space decision, retailer i should determine the corresponding prices for product i in his own store and retailer j 's store as $p_{ii}^N = (\bar{u}_i + c_i)/2$ and $p_{ij}^N = \bar{u}_i - tS_{ji}^N/B_i$, respectively. ■

From the above lemma, we find that, to implement the space-exchange strategy, each retailer allocates a nonzero space to the other retailer. As indicated by Lemma 7, retailer i ($i = 1, 2$) should *fully* use his guest space S_{ji}^N ($j = 1, 2$ and $j \neq i$) to serve some product i consumers who are closer to retailer j 's store, respectively. However, according to our best-response analysis, we note that two retailers do not serve all of their consumers, when they choose the Nash equilibrium. Nevertheless, the two retailers' prices in both host space and guest space are higher than their prices determined when there is no space exchange, as shown in Lemma 5.

We also learn from Lemma 8 that each retailer may or may not set an identical price in two stores. If the space allocated by retailer j to retailer i is $S_{ji}^N = B_i(\bar{u}_i - c_i)/(2t)$, then retailer i 's Nash equilibrium prices in two stores (i.e., p_{ii}^N and p_{ij}^N) will be identical. Because $S_{ji}^N = S_j - B_j(\bar{u}_j - c_j)/(2t)$, we find that $p_{ii}^N = p_{ij}^N$ if $S_j = B_i(\bar{u}_i - c_i)/(2t) + B_j(\bar{u}_j - c_j)/(2t)$. That is, if S_j equals the space that is needed to stock $B_i(\bar{u}_i - c_i)/(2t)$ units of product i and

$B_j(\bar{u}_j - c_j)/(2t)$ units of product j , then retailer i should set an identical price $(\bar{u}_i + c_i)/2$ for product i in two stores. Otherwise, retailer i should determine different prices.

Moreover, two retailers can achieve higher profits compared with those before they exchange shelf space, which means that they should have incentives to cooperate with the space-exchange strategy.

4.2.2 Nash Equilibrium when $\bar{u}_i > c_i + t$ and $\bar{u}_j \leq c_j + t$ ($i, j = 1, 2$ and $i \neq j$)

We now consider the case where all product i ($i = 1, 2$) consumers can achieve a positive net utility from buying in two stores (i.e., $\bar{u}_i > c_i + t$) but some product j ($j = 1, 2$ and $j \neq i$) consumers (e.g., the consumer residing at the site of a retailer's store) cannot draw a positive net utility from their purchases in a store (i.e., $\bar{u}_j \leq c_j + t$).

Lemma 9 When $\bar{u}_i > c_i + t$ and $\bar{u}_j \leq c_j + t$, for $i, j = 1, 2$ and $i \neq j$, there exists a *unique* Nash equilibrium, which depends on the value of S_j , as given in Table 5. ■

Conditions	Nash Equilibrium
$S_j \geq B_i/2 + B_j(\bar{u}_j - c_j)/(2t)$	$S_{ii}^N = B_i/2, S_{jj}^N = B_j(\bar{u}_j - c_j)/(2t)$
$B_i[1 - (\bar{u}_i - c_i)/(2t)] + B_j(\bar{u}_j - c_j)/(2t) \leq S_j \leq B_i/2 + B_j(\bar{u}_j - c_j)/(2t)$	$S_{ii}^N = B_i - S_j + B_j(\bar{u}_j - c_j)/(2t),$ $S_{jj}^N = B_j(\bar{u}_j - c_j)/(2t)$
$S_j \leq B_i[1 - (\bar{u}_i - c_i)/(2t)] + B_j(\bar{u}_j - c_j)/(2t)$	$S_{ii}^N = B_i(\bar{u}_i - c_i)/(2t),$ $S_{jj}^N = B_j(\bar{u}_j - c_j)/(2t)$

Table 5: Nash equilibrium when $\bar{u}_i > c_i + t$ and $\bar{u}_j \leq c_j + t$, for $i, j = 1, 2$ and $i \neq j$.

Similar to our analysis for Case (i) $\bar{u}_i \leq c_i + t$ ($i = 1, 2$), we can use Lemma 3 and 5 to compute two retailers' corresponding optimal prices for each of three possible Nash equilibria in Table 5. We find that each retailer's prices in two stores are both *higher* than the retailer's price before the space exchange. This means that implementing the space-exchange strategy raises the retail prices. From Lemma 9 we find that, if not all product j ($j = 1, 2$ and $j \neq i$) consumers can enjoy a positive net utility, then the total space S_j in retailer j 's store impacts two retailers' Nash equilibrium decisions. Moreover, whatever the value of S_j is, retailer j 's Nash equilibrium space is always determined as $B_j(\bar{u}_j - c_j)/(2t)$. Using Table 5, we obtain Figure 2 to show retailer i 's Nash equilibrium space decision S_{ii}^N as a function of S_j .

If S_j is larger than or equal to $B_i/2 + B_j(\bar{u}_j - c_j)/(2t)$, then retailer j allocates a sufficiently large guest space to retailer i who can then serve a half of product i consumers in retailer j 's

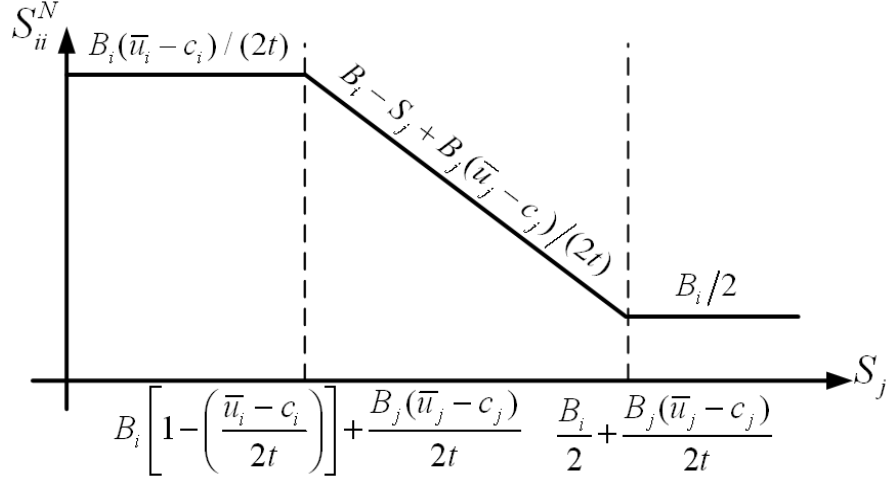


Figure 2: The impact of the space S_j ($j = 1, 2$) on retailer i 's ($i = 1, 2, i \neq j$) Nash equilibrium S_{ii}^N when $\bar{u}_i > c_i + t$ and $\bar{u}_j \leq c_j + t$.

store. According to Figure 2, we find that retailer i should retain his host space S_{ii}^N to serve the other half of his consumers, in order to serve all B_i consumers. However, when S_j is reduced to a value between $B_i[1 - (\bar{u}_i - c_i)/(2t)] + B_j(\bar{u}_j - c_j)/(2t)$ and $B_i/2 + B_j(\bar{u}_j - c_j)/(2t)$, we learn from Figure 2 that retailer i should increase his host space mainly because of the following fact: When S_j decreases, retailer j allocates to retailer i a smaller guest space where retailer i cannot serve a half of product i consumers. In order to fulfill the demands of all B_i consumers in the linear city, retailer i should keep a sufficiently large host space such that all of the consumers who do not decide to buy in retailer j 's store are willing to shop in retailer i 's store.

When S_j is smaller than $B_i[1 - (\bar{u}_i - c_i)/(2t)] + B_j(\bar{u}_j - c_j)/(2t)$, retailer i will obtain a very small guest space and have to serve a small number of product i consumers in retailer j 's store. If retailer i still hopes to fulfill all consumers' demands, then the retailer should keep a large host space to serve most of his consumers. To assure that most consumers (especially those closer to retailer j 's store) are willing to buy in retailer i 's store, retailer i should set a sufficiently low retail price for product i in his host space, which, but, results in a low profit. Therefore, if S_j is significantly small, then retailer i should determine his host space as $S_{ii}^N = B_i(\bar{u}_i - c_i)/(2t)$ in order to guarantee his profit; and as a result, some product i consumers will not decide to buy.

4.2.3 Nash Equilibrium when $\bar{u}_i > c_i + t$ ($i = 1, 2$)

We now solve our non-cooperative game for Case (iii) where both product 1 consumers and product 2 consumers can gain a positive net utility from buying in any store, i.e., $\bar{u}_i > c_i + t$, for $i = 1, 2$. The analysis for this case is much more complicated than the above two cases, and there are many possible Nash equilibria. Thus, we only show the existence of Nash equilibrium in the following theorem but the specific Nash equilibria are provided in online Appendix F.

Lemma 10 When $\bar{u}_i > c_i + t$ ($i = 1, 2$), then the corresponding Nash equilibrium (S_{ii}^N, S_{jj}^N) ($j = 1, 2$ and $j \neq i$) must exist but it may or may not be unique. More specifically, if the total shelf space of two stores is not the same as that needed to exactly serve two retailers' consumers, then Nash equilibrium for the game uniquely exists. Otherwise, the Nash equilibrium may not be unique, which depends on consumers' utilities and trip costs as well as the total available shelf space in each retailer's store. All possible Nash equilibria are given in Table 7 (see online Appendix F). ■

Similar to our analysis for the above cases, we can use Lemma 3 to compute two retailers' corresponding optimal prices for each possible Nash equilibrium given in Table 7. Moreover, we find from Lemma 3 that each retailer's optimal prices in both the host space and the guest space are *higher* than the retailer's price when two retailers do not implement the space-exchange strategy. That is, such a strategy induces two retailers to increase their prices.

5 Further Discussions

In the preceding sections, we analyzed the space-exchange problem and found the Nash equilibrium pricing and space-allocation decisions. We now provide a further discussion on possible changes of our major results in the following two settings: We first consider a more realistic case where there exists at least one "common" consumer who buys both products 1 and 2, and then investigate whether or not our results will change in the setting where a retailer's fixed cost of opening and staffing a new store is considered.

5.1 Presence of Common Consumers

In our model, the set of product 1 consumers and the set of product 2 consumers are assumed to be disjoint. This means that, at any point along the Hotelling line, the product 1 consumer is different from the product 2 consumer. Such an assumption may be applicable to the space-exchange problem to some extent for the following reason: As indicated by the practice of Waitrose and Boots and also by that of Tim Hortons and Cold Stone Creamery, the space-exchange strategy applies *only* when the cooperating retailers' products are neither substitutable nor complementary, e.g., Waitrose's food vs. Boots's healthcare products; and, Tim Hortons's doughnuts vs. Cold Stone Creamery's ice cream. Hence, it should be unusual for any common consumer along the line to *intentionally* buy both products at the same time. For this case, we could regard the common consumer as a "product 1 consumer" and also a "product 2 consumer," who are independent of each other; and then, our existing model could be still used to analyze the space-exchange problem.

Despite the above argument, in reality, there may still exist some common consumer(s) who intend to buy both products concurrently. However, if we relax our assumption on the dependence of the product 1 consumers and the product 2 consumers, then our model may become intractably complicated and we could not draw any meaningful analytical insights. Therefore, we do not incorporate such common consumers into our model but subsequently discuss how our major results would possibly change when a common consumer (who intends to buy both products concurrently) exists.

We learn from our analysis in Sections 3 and 4 that, in the Nash equilibrium, retailer i ($i = 1, 2$) may serve all of B_i consumers or may serve only a part of those consumers. Next, we provide our discussion for three cases: (a) both retailers i and j ($j = 1, 2$ and $j \neq i$) serve all of their consumers; (b) both retailers i and j do not serve all of their consumers; and (c) retailer i serves all of B_i consumers but retailer j does not serve all of B_j consumers. For each case, we discuss a representative situation, as shown in Figure 3, where (a), (b), and (c) represent Cases (a), (b), and (c), respectively. [Our discussions on other situations in each case are similar to our discussion for the representative situation.]

We begin by discussing the impacts of the presence of a common consumer in Case (a), which corresponds to Figure 3(a) where the product i ($i = 1, 2$) consumers locating at the left of \tilde{x}_i buy in retailer 1's store and those consumers at the right of \tilde{x}_i buy in retailer 2's store.

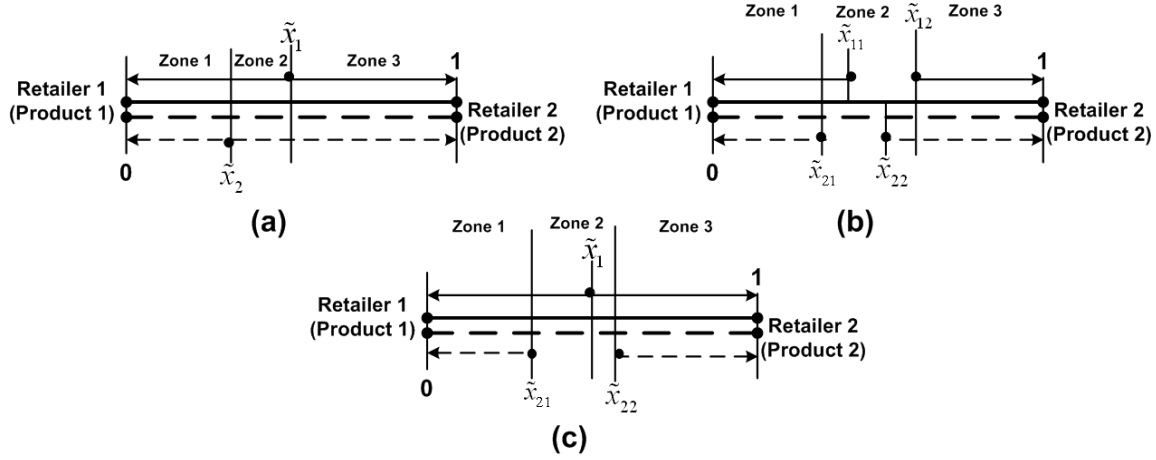


Figure 3: The impacts of the presence of a common consumer on major results for three representative situations.

As Figure 3(a) indicates, the product 1 and the product 2 consumers in zone 1 visit retailer 1’s store to buy products 1 and 2, respectively, when there is no common consumer. If we assume that there is a common consumer at a point in zone 1, then the consumer will still buy both products in retailer 1’s store. That is, for Case (a), the presence of common consumers in zone 1 would not increase the demands for two products in each store; thus, it may not result in any change in two retailers’ pricing and space-allocation decisions. Similarly, any common consumer in zone 3 will decide to buy two products in retailer 2’s store. This does not increase or decrease the demands faced by two retailers in each store, and would not change two retailers’ decisions.

If a common consumer is located in zone 2, then the total demand for each product in two stores should be unchanged but the demands faced by two retailers in each store may differ from those with no common consumer. Specifically, if there is no common consumer in zone 2, then the product 1 and the product 2 consumers will buy in different stores. But, if a common consumer in zone 2 is closer to \tilde{x}_2 , then he or she may be likely to buy both products in retailer 1’s store. As a result, compared with the “no common consumer” case, the demand for product 2 in retailer 1’s store is increased by 1 unit and that in retailer 2’s store is decreased by 1 unit, whereas the demand for product 1 in each store is not changed. Even though retailer 2 needs to sell one more unit in retailer 1’s store, retailer 1 is unlikely to allocate one more space to retailer 2 because retailer 1 should keep his host space to serve existing customers. If two retailers have already used the total space S_1 , then retailer 2 may respond by increasing his price p_{21} for product 2 in retailer 1’s store but decreasing his price p_{22} for product 2 in his own store, in order to “move” a consumer from retailer 1’s store to retailer 2’s store. On the other

hand, if there is an excess space in retailer 1's store, then retailer 2 should take over one more unit of the space to satisfy the demand by the common consumer; as a result, two retailers do not change their pricing decisions.

Next, we discuss the impacts of the presence of a common consumer in Case (b), which corresponds to Figure 3(b) where the product i ($i = 1, 2$) consumers residing at the left of \tilde{x}_{i1} and those at the right of \tilde{x}_{i2} buy in retailer 1's and retailer 2's stores, respectively. Using our arguments for Case (a), a common consumer in zones 1 and 3 would have no impact on two retailers' decisions; and, the presence of a common consumer in zone 2 would lead to an increase in the demand faced by one or two retailers at a store. The retailers may change their prices at two stores if there is no excess space at the store where the demand rises, or may keep the prices unchanged and use the excess space otherwise. Similarly, we find that, for Case (c), there would be no change if a common consumer is in zones 1 and 3, but the existence of a common consumer in zone 2 would result in the price changes at two stores when there is no excess space. Summarizing our above discussion, we draw the implications as given in the following remark.

Remark 3 If a consumer intends to buy both products 1 and 2, then the demand faced by one or two retailers at a store may be increased and two retailers may respond by changing their pricing and space-allocation decisions. Specifically, the presence of a common consumer close to a store is unlikely to change two retailers' decisions. However, if a common consumer resides in a middle point between two stores, then one or two retailers may face an increasing demand at a store, thereby increasing their prices if there is no excess space at the store or keeping the prices unchanged but using the excess space. As a result, if a common consumer exists, then two retailers' profits could be increased. Moreover, our above discussion also implies that Propositions 1, 2, and 3 should hold in the presence of common consumers. \triangleleft

5.2 Presence of Fixed Costs

In actual practice, two retailers (e.g., Waitrose and Boots, Tim Hortons and Cold Stone Creamery) may be willing to exchange shelf space instead of opening their own new stores. A motivation for two retailers to exchange shelf space would be mainly attributed to the fact that each retailer incurs a fixed cost in opening and staffing a new store, but does not pay for such a cost in exchanging his shelf space with the other retailer. We now examine whether or not

our major results would change if we consider fixed costs in our Hotelling model. Suppose that retailer i ($i = 1, 2$) will decide to (i) open a new store at retailer j 's site ($j = 1, 2$ and $j \neq i$), or (ii) exchange shelf space with retailer j . Each retailer should choose one from the two options (i) and (ii).

Noting that two retailers' fixed costs are independent of their pricing and space-allocation decisions, we find that, if each retailer's store is sufficiently large, then incorporating such costs into our model should not change two retailers' decisions, and exchanging shelf space should result in a higher profit for each retailer compared with opening a new store. Otherwise, if a retailer's store is small, then the retailer cannot allot a sufficiently large space to the other retailer, who may then respond by opening a new store instead of exchanging shelf space. Such a result is in agreement with Proposition 3, which indicates that two retailers should decide to implement the space-exchange strategy in Nash equilibrium, if and only if each retailer's total shelf space is large enough to serve more than a half of his consumers. In Section 4 we perform our game analysis, assuming that each retailer has a sufficiently large store. Such an assumption is compatible with the practice that the retailers exchanging shelf space include, e.g., Waitrose, Boots, Tim Hortons, and Cold Stone Creamery. It thus follows that our results in this paper do not change if we consider each retailer's fixed cost of opening and staffing a new store.

6 Summary and Concluding Remarks

This paper is motivated by the practice of Waitrose and Boots (and also, Tim Hortons and Cold Stone Creamery) where these retailers exchange shelf space to increase their profits. We use the Hotelling model to analyze a two-retailer problem. Before the space exchange, each consumer can buy only in one store; but, after two retailers implement the space-exchange strategy, the consumer can access each retailer's product in two stores and thus visit a store closer to the consumer's location to buy.

We first assume that two retailers do not exchange shelf space, and maximize each retailer's profit to find the optimal price for the retailer's product in his own store. Then, we determine each retailer's optimal prices in two stores given his host space and guest space under the space-exchange strategy, and find that the space-dependent prices are impacted by whether or not all of the retailer's consumers can enjoy a positive net utility from buying in any store.

Using the optimal space-dependent price, we consider a non-cooperative game where each retailer makes the space allocation decision for his own store to maximize the total profit in two stores. We show that two retailers should decide to implement the space-exchange strategy in the game, if and only if the total space in each retailer's store is large enough to serve more than a half of the retailer's consumers. Nash equilibrium for the game may or may not uniquely exist, depending on consumers' utilities and trip costs as well as the total space in each store. We also find that, in the Nash equilibrium, each retailer's prices in two stores may or may not be identical but they are both higher than the retailer's price before the space exchange, and two retailers' profits are higher than those before they implement the space-exchange strategy. We also discuss possible changes of our major results when there exists a common consumer who buys both products 1 and 2, and those when a retailer's fixed cost for opening and staffing a new store is considered.

As we discuss in Section 1, the informational advantage and the risk reduction should be the two main advantages of the space-exchange strategy. In this paper, we focus on consumers' increased choices and reduced trip costs, which should be the major advantage for the strategy. The analysis of the informational advantage and the risk reduction for the space-exchange problem would be a future research direction.

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Online Supplements
“The Retail Space-Exchange Problem with Pricing and Space Allocation
Decisions”

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Appendix A Proofs of Lemmas

Proof of Lemma 1. We first consider retailer 1’s optimal pricing decision. We learn from (2) that retailer 1’s profit depends on the comparison between $x_1 = (\bar{u}_1 - p_1)/t$ and S_1/B_1 . We perform our analysis for the following two cases:

1. When $S_1 \geq B_1$, retailer 1’s profit function in (2) can be re-written as $\pi_1 = B_1 \times (p_1 - c_1) \times x_1 = B_1 \times (p_1 - c_1) \times (\bar{u}_1 - p_1)/t$. The first- and second-order derivatives of π_1 w.r.t. p_1 are thus computed as,

$$\frac{\partial \pi_1}{\partial p_1} = \frac{B_1 \times (\bar{u}_1 - 2p_1 + c_1)}{t} \quad \text{and} \quad \frac{\partial^2 \pi_1}{\partial p_1^2} = -\frac{2B_1}{t}.$$

Temporarily ignoring the constraint that $\bar{u}_1 - t \leq p_1 \leq \bar{u}_1$, we find that the optimal price maximizing π_1 is $(\bar{u}_1 + c_1)/2$, which is smaller than or equal to \bar{u}_1 because $\bar{u}_1 \geq c_1$. Considering the constraint that $\bar{u}_1 - t \leq p_1 \leq \bar{u}_1$, we find the optimal price for this case as $p_1^* = \max[\bar{u}_1 - t, (\bar{u}_1 + c_1)/2]$.

Next, we calculate retailer 1’s maximum profit. From the above, we find that we should compare $(\bar{u}_1 + c_1)/2$ and $\bar{u}_1 - t$ to determine retailer 1’s optimal price and compute the corresponding maximum profit.

- (a) If $(\bar{u}_1 + c_1)/2 \geq \bar{u}_1 - t$, or, $c_1 + 2t \geq \bar{u}_1$, then retailer 1’s optimal price is $p_1^* = (\bar{u}_1 + c_1)/2$ and his maximum profit is calculated as, $\pi_1 = B_1 \times [(\bar{u}_1 + c_1)/2 - c_1] \times [\bar{u}_1 - (\bar{u}_1 + c_1)/2]/t = B_1 \times (\bar{u}_1 - c_1)^2/(4t)$.
 - (b) If $c_1 + 2t \leq \bar{u}_1$, then retailer 1’s optimal price is $p_1^* = \bar{u}_1 - t$ and his maximum profit is found as $\pi_1 = B_1 \times (\bar{u}_1 - t - c_1)$.
2. When $S_1 < B_1$, we need to compare x_1 and S_1/B_1 to determine the optimal retail price for this case. Specifically,

- (a) When $x_1 \leq S_1/B_1$, or, $p_1 \geq \bar{u}_1 - tS_1/B_1$, retailer 1’s profit function can be re-written as $\pi_1 = B_1 \times (p_1 - c_1) \times (\bar{u}_1 - p_1)/t$. Noting that $S_1/B_1 < 1$, we find that $\bar{u}_1 - tS_1/B_1 > \bar{u}_1 - t$ and the maximization constraint for this case thus becomes $\bar{u}_1 - tS_1/B_1 \leq p_1 \leq \bar{u}_1$. Using our argument for the first case, we find the optimal price for this case as $\max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - tS_1/B_1]$.
- (b) When $x_1 \geq S_1/B_1$, or, $p_1 \leq \bar{u}_1 - tS_1/B_1$, retailer 1’s profit function can be re-written as $\pi_1 = B_1 \times (p_1 - c_1) \times S_1/B_1$, which is increasing in p_1 . Therefore, for this case, the optimal retail price is $\bar{u}_1 - tS_1/B_1$.

According to the above, we find that, when $S_1 < B_1$, retailer 1’s optimal price is determined as $p_1^* = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - tS_1/B_1]$. Next, we calculate retailer 1’s maximum profit.

- (a) If $(\bar{u}_1 + c_1)/2 \geq \bar{u}_1 - tS_1/B_1$, or, $c_1 + 2tS_1/B_1 \geq \bar{u}_1$, then retailer 1’s optimal price is $p_1^* = (\bar{u}_1 + c_1)/2$, and his maximum profit is calculated as, $\pi_1 = B_1 \times [(\bar{u}_1 + c_1)/2 - c_1] \times [\bar{u}_1 - (\bar{u}_1 + c_1)/2]/t = B_1 \times (\bar{u}_1 - c_1)^2/4$.

- (b) If $c_1 + 2tS_1/B_1 \leq \bar{u}_1$, then retailer 1's optimal price is $p_1^* = \bar{u}_1 - tS_1/B_1$ and his maximum profit is $\pi_1 = (\bar{u}_1 - tS_1/B_1 - c_1) \times S_1$.

We can similarly compute retailer 2's optimal price and maximum profit. The lemma is thus proved. ■

Proof of Lemma 2. We first perform our analysis for retailer 1. As Remark 1 indicates, there are two cases in which retailer 1 can set his prices in two stores to affect consumers' demands. We compute the retailer's optimal prices for the two cases as follows:

1. Retailer 1 determines his retail prices (p_{11}, p_{12}) under the constraint that $p_{21} + p_{11} \leq 2\bar{u}_1 - t$. As discussed in Remark 1, for this case, all consumers must have a non-negative utility and thus decide to buy from retailer 1. Because the demands faced by retailer 1 in two stores are $D_{11} = B_1(p_{12} - p_{11} + t)/(2t)$ and $D_{12} = B_1(p_{11} - p_{12} + t)/(2t)$, we can construct retailer 1's profit for this case as,

$$\pi_1 = B_1(p_{11} - c_1) \frac{p_{12} - p_{11} + t}{2t} + B_1(p_{12} - c_1) \frac{p_{11} - p_{12} + t}{2t}. \quad (4)$$

Retailer 1's maximization problem is thus developed as, $\max \pi_1$, s.t. $p_{12} + p_{11} \leq 2\bar{u}_1 - t$ and $-t \leq p_{12} - p_{11} \leq t$. Note that the second constraint is involved because, as discussed previously, all of B_1 consumers are uniformly distributed between the sites of retailers 1 and 2.

The first-, second-order, and cross-partial derivatives of π_1 w.r.t. p_{11} and p_{12} are calculated as,

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_{11}} &= B_1 \frac{2(p_{12} - p_{11}) + t}{2t}, & \frac{\partial^2 \pi_1}{\partial p_{11}^2} &= -\frac{B_1}{t} < 0; \\ \frac{\partial \pi_1}{\partial p_{12}} &= B_1 \frac{2(p_{11} - p_{12}) + t}{2t}, & \frac{\partial^2 \pi_1}{\partial p_{12}^2} &= -\frac{B_1}{t} < 0; & \frac{\partial^2 \pi_1}{\partial p_{21} \partial p_{11}} &= \frac{B_1}{t} > 0. \end{aligned}$$

It is easy to show the Hessian's negative definiteness. Thus, the profit π_1 in (4) is jointly concave in the prices p_{11} and p_{12} . Setting the first-order derivatives to zero and solving them, we have the optimal retail prices for the first case as, $p_{12} = p_{11} = (2\bar{u}_1 - t)/2 = \bar{u}_1 - t/2$, which satisfy the constraints that $p_{12} + p_{11} + t \leq 2\bar{u}_1$ and $-t \leq p_{12} - p_{11} \leq t$. The corresponding demands in retailer 1's own store and retailer 2's store are then calculated as $D_{11} = D_{12} = B_1/2$, respectively. Moreover, retailer 1's maximum profits in the two stores are both obtained as $B_1(\bar{u}_1 - c_1 - t/2)/2$; and, the retailer's total profit is computed as $\pi_1 = B_1(\bar{u}_1 - c_1 - t/2)$.

2. Retailer 1 determines his prices p_{11} and p_{12} such that $p_{11} + p_{12} \geq 2\bar{u}_1 - t$. We learn from Remark 1 that, for this case, some consumer(s) may not buy from retailer 1, who may thus partially satisfy the demand $B_1(2\bar{u}_1 - p_{11} - p_{12})/t$. The demands for product 1 in two stores are $D_{11} = B_1(\bar{u}_1 - p_{11})/t$ and $D_{12} = B_1(\bar{u}_1 - p_{12})/t$; it then follows that retailer 1's profit is computed as,

$$\pi_1 = B_1(p_{11} - c_1) \frac{\bar{u}_1 - p_{11}}{t} + B_1(p_{12} - c_1) \frac{\bar{u}_1 - p_{12}}{t}. \quad (5)$$

Retailer 1 should find his optimal retail prices by solving the following maximization problem: $\max \pi_1$, s.t. $p_{11} + p_{12} + t \geq 2\bar{u}_1$ and $-t \leq p_{12} - p_{11} \leq t$. The first-, second-order,

and cross-partial derivatives of π_1 w.r.t. p_{11} and p_{12} are calculated as,

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_{11}} &= B_1 \frac{\bar{u}_1 + c_1 - 2p_{11}}{t}, & \frac{\partial^2 \pi_1}{\partial p_{11}^2} &= -\frac{2B_1}{t} < 0; \\ \frac{\partial \pi_1}{\partial p_{12}} &= B_1 \frac{\bar{u}_1 + c_1 - 2p_{12}}{t}, & \frac{\partial^2 \pi_1}{\partial p_{12}^2} &= -\frac{2B_1}{t} < 0; & \frac{\partial^2 \pi_1}{\partial p_{12} \partial p_{11}} &= 0. \end{aligned}$$

It is easy to prove that the Hessian's definiteness is negative, and retailer 1's profit π_1 in (5) is jointly concave in p_{11} and p_{12} . However, we cannot immediately compute the optimal prices by solving the first-order conditions (i.e., $\partial \pi_1 / \partial p_{11} = 0$ and $\partial \pi_1 / \partial p_{12} = 0$), because, otherwise, the demand in terms of optimal prices *may* be greater than the total demand B_1 , which is specified as follows:

Temporarily ignoring the above concern regarding the demand, we solve the equations that $\partial \pi_1 / \partial p_{11} = 0$ and $\partial \pi_1 / \partial p_{12} = 0$ to make retailer 1's optimal pricing decisions as $p_{11} = p_{12} = (\bar{u}_1 + c_1)/2$. The resulting demands in two stores are thus computed as $D_{11} = D_{12} = B_1(\bar{u}_1 - c_1)/(2t)$; the total demand is $D_{11} + D_{12} = B_1(\bar{u}_1 - c_1)/t$, which may be greater than B_1 , depending on the values of \bar{u}_1 , c_1 , and t . Next, we consider the comparison between \bar{u}_1 and $c_1 + t$ to determine retailer 1's optimal prices.

- (a) If $\bar{u}_1 \leq c_1 + t$, then $D_{11} + D_{12} \leq B_1$; this means that retailer 1 should adopt the optimal prices $p_{11} = p_{12} = (\bar{u}_1 + c_1)/2$, which satisfy the constraints that $p_{11} + p_{12} \geq 2\bar{u}_1 - t$ and $-t \leq p_{12} - p_{11} \leq t$. The retailer may thus fulfill the demands of some rather than all product 1 consumers. As a result, retailer 1's profits in two stores are both calculated as $B_1(\bar{u}_1 - c_1)^2/(4t)$, and his total profit is $\pi_1 = B_1(\bar{u}_1 - c_1)^2/(2t)$.
- (b) If $\bar{u}_1 > c_1 + t$, then $D_{11} + D_{12} > B_1$, which implies that the optimal prices are too low and retailer 1 can raise his prices to increase his profit margin without losing any consumer. Since retailer 1 will satisfy all B_1 consumers, similar to Case 1, we can find his decisions as $p_{11} = p_{12} = \bar{u}_1 - t/2$, which is higher than the price $(\bar{u}_1 + c_1)/2$ (that is optimal when $\bar{u}_1 \leq c_1 + t$). The demands in two stores are then computed as $D_{11} = D_{12} = B_1/2$, and the total demand is $D_{11} + D_{12} = B_1$. Retailer 1's profits in two stores are both computed as $B_1(\bar{u}_1 - c_1 - t/2)$, and his total profit is thus $\pi_1 = B_1(\bar{u}_1 - c_1 - t/2)$.

Note that, if the retail prices p_{11} and p_{12} are both reduced to retailer 1's acquisition cost c_1 and the product 1 consumer residing at the location 1 (i.e., retailer 2's store) intends to buy in retailer 1's store, then the consumer enjoys the utility \bar{u}_1 but incurs the cost $c_1 + t$. The above condition that $\bar{u}_1 \leq c_1 + t$ means that the product 1 consumer at the site of retailer 2's store has a non-positive net utility if he or she goes to retailer 1's store to buy a unit of product 1.

We can similarly perform our analysis for retailer 2. This proves the lemma. ■

Proof of Lemma 3. We first analyze retailer 1's optimal pricing decisions. We consider two cases—i.e., $S_{11} + S_{21} \geq B_1$ and $S_{11} + S_{21} < B_1$ —to determine retailer 1's optimal pricing decisions. We first investigate the case that $S_{11} + S_{21} \geq B_1$, which means that retailer 1's total space $S_{11} + S_{21}$ in two stores is large enough to stock B_1 units of products. When $S_{11} + S_{21} \geq B_1$, we need to consider the following three scenarios: (i) $S_{11} \geq B_1/2$ and $S_{21} \geq B_1/2$; (ii) $S_{11} < B_1/2$ and $S_{21} > B_1/2$; and (iii) $S_{11} > B_1/2$ and $S_{21} < B_1/2$. For the scenario (i), we learn from Lemma 2 that the optimal prices are determined as $p_{11} = p_{12} = \bar{u}_1 - t/2$, and the

resulting demands in two stores are both equal to $B_1/2$.

Next we consider the scenario (ii), where, in retailer 1's own store, retailer 1 cannot satisfy a half of the total demand. This means that, even though the retailer can carry sufficient number of product 1 in two stores, the retailer's optimal pricing decisions under the capacity constraint may not result in the fulfillment of the total demand B_1 . Recall from Remark 1 that retailer 1 may determine his prices such that $p_{11} + p_{12} \leq 2\bar{u}_1 - t$ or may make his pricing decisions such that $p_{11} + p_{12} \geq 2\bar{u}_1 - t$. Next, we maximize retailer 1's profit when $p_{11} + p_{12} \leq 2\bar{u}_1 - t$ and maximize that when $p_{11} + p_{12} \geq 2\bar{u}_1 - t$, and then compare the maximum profits to find the optimal prices for the retailer.

1. If $p_{11} + p_{12} \leq 2\bar{u}_1 - t$, then, according to Remark 1, all product 1 consumers should buy from retailer 1 in either retailer 1's store or retailer 2's store; and, the demands faced by retailer 1 in two stores are $D_{11} = B_1(p_{12} - p_{11} + t)/(2t)$ and $D_{12} = B_1(p_{11} - p_{12} + t)/(2t)$. Temporarily ignoring the capacity (space) constraint, we develop the retailer's profit function as,

$$\begin{aligned}\pi_1 &= B_1(p_{11} - c_1)\frac{p_{12} - p_{11} + t}{2t} + B_1(p_{12} - c_1)\frac{p_{11} - p_{12} + t}{2t} \\ &= -B_1\frac{(p_{12} - p_{11})^2}{2t} + B_1\frac{p_{11} + p_{12}}{2} - B_1c_1,\end{aligned}$$

which is decreasing in the difference between p_{12} and p_{11} but increasing in $p_{11} + p_{12}$. This means that retailer 1 should consider the following two strategies to make his optimal pricing decisions: (i) The difference between retailer 1's prices in two stores should be as small as possible; and (ii) The value $p_{11} + p_{12}$ should be as large as possible. Since p_{11} and p_{12} should be determined such that $p_{11} + p_{12} \leq 2\bar{u}_1 - t$, the optimal prices should satisfy the equality that $p_{11} + p_{12} = 2\bar{u}_1 - t$. Because the difference between p_{12} and p_{11} should be as small as possible, the difference between D_{11} and D_{12} should be minimized under the constraints that $S_{11} < B_1/2$ and $S_{21} > B_1/2$. Therefore, the retailer needs to determine his prices such that the total demand for product 1 in retailer 1's and retailer 2's stores are equal to S_{11} and $B_1 - S_{11}$, respectively. That is, solving the following equations:

$$\begin{cases} B_1(p_{12} - p_{11} + t)/(2t) = S_{11}, \\ p_{11} + p_{12} = 2\bar{u}_1 - t, \end{cases}$$

we can find the retailer's prices as $p_{11} = \bar{u}_1 - tS_{11}/B_1$ and $p_{12} = \bar{u}_1 - t(1 - S_{11}/B_1)$, which satisfies the constraint that $-t \leq p_{12} - p_{11} \leq t$.

2. If retailer 1's pricing decisions are made such that $p_{11} + p_{12} \geq 2\bar{u}_1 - t$, then some consumer(s) may not buy from retailer 1, who may thus only satisfy a part of the total demand B_1 . The retailer's profits generated in retailer 1's and retailer 2's stores are written as,

$$(p_{11} - c_1) \times \min[S_{11}, B_1(\bar{u}_1 - p_{11})/t] \quad \text{and} \quad (p_{12} - c_1) \times \min[B_1 - S_{11}, B_1(\bar{u}_1 - p_{12})/t].$$

According to Lemma 2, the optimal price p_{11} maximizing $B_1(p_{11} - c_1)(\bar{u}_1 - p_{11})/t$ should be equal to $(\bar{u}_1 + c_1)/2$, and the resulting demand for product 1 in retailer 1's store is $B_1(\bar{u}_1 - c_1)/(2t)$, which is greater than $B_1/2$ because $\bar{u}_1 > c_1 + t$. Since $S_{11} < B_1/2$, the retailer should increase the price p_{11} from $(\bar{u}_1 + c_1)/2$ to a value such that the total

demand in retailer 1's store—i.e., $B_1(\bar{u}_1 - p_{11})/t$ —is equal to the available number S_{11} . Solving the equation that $B_1(\bar{u}_1 - p_{11})/t = S_{11}$ gives the price $p_{11} = \bar{u}_1 - tS_{11}/B_1$, which is greater than $(\bar{u}_1 + c_1)/2$ since $\bar{u}_1 > c_1 + t$.

Because of the constraint that $p_{11} + p_{12} + t \geq 2\bar{u}_1$, the price p_{12} should be greater than or equal to $\bar{u}_1 - t(1 - S_{11}/B_1)$, which occurs if and only if the total demand for product 1 in retailer 2's store—that is, $B_1(\bar{u}_1 - p_{12})/t$ —is smaller than or equal to the available quantity $B_1 - S_{11}$. Therefore, retailer 1's profit generated in retailer 2's store is calculated as $B_1(p_{12} - c_1)(\bar{u}_1 - p_{12})/t$, which should be maximized subject to $p_{12} \geq \bar{u}_1 - t(1 - S_{11}/B_1)$ and $-t \leq p_{12} - p_{11} \leq t$. Lemma 2 indicates that the optimal price p_{12} maximizing $B_1(p_{12} - c_1)(\bar{u}_1 - p_{12})/t$ is $(\bar{u}_1 + c_1)/2$. It thus follows that the optimal price p_{12} is determined as $p_{12} = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - t(1 - S_{11}/B_1)]$. Because $S_{11} < B_1/2$ and $p_{12} \geq \bar{u}_1 - t(1 - S_{11}/B_1)$, we find that $p_{12} - p_{11} \geq -t(1 - 2S_{11}/B_1) > -t$. Note from the above that $(\bar{u}_1 + c_1)/2 < \bar{u}_1 - tS_{11}/B_1$; thus, $p_{12} - p_{11} \leq 0 < t$. Hence, the constraint that $-t \leq p_{12} - p_{11} \leq t$ is satisfied.

In conclusion, we find that, if $S_{11} + S_{21} \geq B_1$ but $S_{11} < B_1/2$, retailer 1's optimal prices in his own store and retailer 2's store are $p_{11} = \bar{u}_1 - tS_{11}/B_1$ and $p_{12} = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - t(1 - S_{11}/B_1)]$. Similarly, if $S_{11} + S_{21} \geq B_1$ but $S_{21} < B_1/2$, retailer 1's optimal prices in his own store and retailer 2's store are $p_{11} = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - t(1 - S_{21}/B_1)]$ and $p_{12} = \bar{u}_1 - tS_{21}/B_1$.

Next, we compute retailer 1's optimal prices when the retailer's total space $S_{11} + S_{21}$ cannot stock B_1 units of product 1, i.e., $S_{11} + S_{21} < B_1$. For this scenario, the retailer's profits in retailer 1's and retailer 2's stores are $(p_{11} - c_1) \times \min[S_{11}, B_1(\bar{u}_1 - p_{11})/t]$ and $(p_{12} - c_1) \times \min[S_{21}, B_1(\bar{u}_1 - p_{12})/t]$. If $p_{11} \leq \bar{u}_1 - tS_{11}/B_1$, then retailer 1's profit in his own store becomes $B_1(p_{11} - c_1) \times S_{11}$, which is increasing in p_{11} . Therefore, the optimal price p_{11} maximizing $B_1(p_{11} - c_1) \times S_{11}$ is equal to $\bar{u}_1 - tS_{11}/B_1$. On the other hand, if $p_{11} \geq \bar{u}_1 - tS_{11}/B_1$, then retailer 1's profit in his own store becomes $B_1(p_{11} - c_1) \times (\bar{u}_1 - p_{11})/t$, which is maximized at the point $p_{11} = (\bar{u}_1 + c_1)/2$. Therefore, retailer 1's optimal price p_{11} is obtained as $p_{11} = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - tS_{11}/B_1]$. Similarly, the retailer's optimal price p_{12} is obtained as $p_{12} = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - tS_{21}/B_1]$.

Similar to the above analysis for retailer 1, we can find retailer 2's optimal prices. This proves the lemma. ■

Proof of Lemma 4. We first consider the price comparison for retailer 1, which will similarly apply to retailer 2. We find from Lemma 1 that, when the two retailers do not exchange shelf space, retailer 1's optimal price p_1^* depends on S_1 ; and we note from Lemma 3 that retailer 1's optimal price p_{11}^* and p_{12}^* under the space-exchange strategy depend on the values of S_{11} and S_{21} . Thus, for this proof, we have to compare any pair of optimal prices in two settings.

1. If $S_{11} + S_{21} \geq B_1$, $S_{11} \geq B_1/2$, and $S_{21} \geq B_1/2$, then p_{11}^* and p_{12}^* are obviously greater than $\bar{u}_1 - t$. We also find that $\bar{u}_1 - t/2 > (\bar{u}_1 + c_1)/2$ because $\bar{u}_1 > c_1 + t$. Therefore, $p_{11}^* = p_{12}^* > p_1^* = \max((\bar{u}_1 + c_1)/2, \bar{u}_1 - t)$, which applies when $S_1 \geq B_1$. We then consider the comparison when $S_1 < B_1$. Because $S_1 \geq S_{11} \geq B_1/2$, $\bar{u}_1 - tS_1/B_1 < \bar{u}_1 - t/2 = p_{11}^* = p_{12}^*$. That is, for this case, $p_{11}^* = p_{12}^* > p_1^*$.
2. If $S_{11} + S_{21} \geq B_1$, $S_{11} < B_1/2$, and $S_{21} > B_1/2$, then we can easily show that $p_{11}^* > p_1^*$. However, p_{12}^* may or may not be greater than p_1^* . More specifically, when $S_1 \geq B_1$, $p_{12}^* > p_1^*$; but, when $S_1 < B_1$, p_{12}^* may or may not be greater than p_1^* . If $S_1 \geq B_1(\bar{u}_1 - c_1)/(2t)$, then, as Lemma 1 indicates, p_1^* is equal to $(\bar{u}_1 + c_1)/2$ or $\bar{u}_1 - t$, which is smaller than p_{12}^* . Otherwise, if $S_1 < B_1(\bar{u}_1 - c_1)/(2t)$, then $p_1^* = \bar{u}_1 - tS_1/B_1$. Therefore, a sufficient condition under which $p_{12}^* \geq p_1^*$ can be found as $\bar{u}_1 - t(1 - S_{11}/B_1) \geq \bar{u}_1 - tS_1/B_1$, or,

- $S_{11} + S_1 \geq B_1$.
3. If $S_{11} + S_{21} \geq B_1$, $S_{11} > B_1/2$, and $S_{21} < B_1/2$, then we find that $p_{12}^* > p_1^*$. Moreover, because $S_1 + S_{21} \geq S_{11} + S_{21} \geq B_1$, $1 - S_{21}/B_1 \leq S_1/B_1$ and $p_{11}^* \geq p_1^*$.
 4. If $S_{11} + S_{21} < B_1$, then $p_{11}^* > p_1^*$; but, p_{12}^* may or may not be greater than p_1^* . Similar to the above proof for item 2, we find that, for this case, a sufficient condition under which $p_{12}^* \geq p_1^*$ is found as, $S_{21} \leq S_1$.

Similarly, we can compare the prices for retailer 2. This proves the lemma. ■

Proof of Lemma 5. We consider the decisions for retailer 1 in this proof, since those for retailer 2 are similar. If $S_{11} \geq B_1(\bar{u}_1 - c_1)/2t$ and $S_{21} \geq B_1(\bar{u}_1 - c_1)/2t$, then retailer 1 should not serve all product 1 consumers, as suggested by Lemma 2, and his price is determined as $p_{11} = p_{12} = (\bar{u}_1 + c_1)/2$.

However, if the space allocated to retailer 1 in his own store is smaller than $B_1(\bar{u}_1 - c_1)/(2t)$, i.e., $S_{11} < B_1(\bar{u}_1 - c_1)/(2t)$, then the retailer should increase his price p_{11} from $(\bar{u}_1 + c_1)/2$ to the value that can be obtained by solving the equation that $B_1(\bar{u}_1 - p_{11})/t = S_{11}$. That is, if $S_{11} < B_1(\bar{u}_1 - c_1)/(2t)$, then retailer 1's price should be determined as $p_{11} = \bar{u}_1 - tS_{11}/B_1$, which is greater than $(\bar{u}_1 + c_1)/2$ because $S_{11} < B_1(\bar{u}_1 - c_1)/(2t)$. Note that, if $S_{11} \geq B_1(\bar{u}_1 - c_1)/2t$, then $(\bar{u}_1 + c_1)/2 \geq \bar{u}_1 - tS_{11}/B_1$. Therefore, we conclude that, when $\bar{u}_1 \leq c_1 + t$, retailer 1's price for product 1 in his own store is $p_{11} = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - tS_{11}/B_1]$. Similarly, the retail price in retailer 2's store is obtained as $p_{12} = \max[(\bar{u}_1 + c_1)/2, \bar{u}_1 - tS_{21}/B_1]$. For the price comparison, see the proof of Lemma 4. This proves the lemma. ■

Proof of Lemma 6. In this proof, we consider retailer 1's best-response decisions. Retailer 2's decisions similarly follow. From Lemmas 3 and 5 and Table 3, we find that retailer 1's optimal pricing decision and corresponding maximum profit depend on the values of S_{11} and S_{21} . Note that retailer 1's guest space S_{21} may be greater than, equal to, or smaller than $B_1/2$; and, retailer 1's decision S_{11} cannot exceed his total space S_1 , which may be greater than or may be smaller than $B_1/2$. Therefore, we have to consider the following scenarios:

1. If $S_1 + S_{21} \geq B_1$, then retailer 1 is able to serve all of B_1 product 1 consumers. However, the optimal value of S_{11} depends on the value of S_{21} , as indicated by Table 3. We consider two possibilities: $S_{21} \geq B_1/2$ and $S_{21} < B_1/2$. If $S_{21} \geq B_1/2$, then we learn from Table 3 that, when $S_{11} \geq B_1/2$, retailer 1's maximum profit is $\pi_1^1 \equiv B_1(\bar{u}_1 - c_1 - t/2)$; when $S_{11} < B_1/2$, the retailer's maximum profit is π_1^2 or π_1^3 , which depends on the comparison between \bar{u}_1 and $c_1 + 2t(1 - S_{11}/B_1)$.

Temporarily ignoring the value of S_1 , we learn from the proof of Corollary 1 that $\pi_1^3 \leq \pi_1^2 < \pi_1^1$; and, to maximize π_1^3 , S_{11} must be determined as $S_{11} = B_1[1 - (\bar{u}_1 - c_1)/(2t)]$. Note that $\pi_1^3 = \pi_1^2$ when $S_{11} = B_1[1 - (\bar{u}_1 - c_1)/(2t)]$. Hence, if $S_1 + S_{21} \geq B_1$ and $S_{21} \geq B_1/2$, then retailer 1's optimal decision is given as follows:

- (a) If $S_1 \geq B_1/2$, then the retailer's optimal host space is $S_{11}^* = B_1/2$ and the space allocated to retailer 2 is calculated as $S_{12}^* = S_1 - B_1/2$. In addition, the retailer should only use the guest space $B_1/2$ for his sale in retailer 2's store. All of B_1 product 1 consumers are served when $S_1 \geq B_1/2$.
- (b) If $S_1 < B_1/2$, then the retailer's optimal decision is dependent on the comparison between S_1 and $B_1[1 - (\bar{u}_1 - c_1)/(2t)]$. More specifically, if $B_1[1 - (\bar{u}_1 - c_1)/(2t)] \leq S_1 < B_1/2$, then retailer 1's profit π_1^2 is increasing in S_{11} . Therefore, his optimal host space is $S_{11}^* = S_1$, which means that the retailer does not allocate any space to

retailer 2. Retailer 1 should accept the guest space $(B_1 - S_1)$, which is smaller than or equal to S_{21} because $S_1 + S_{21} \geq B_1$. All of B_1 product 1 consumers are served when $B_1[1 - (\bar{u}_1 - c_1)/(2t)] \leq S_1 < B_1/2$.

Otherwise, if $S_1 < B_1[1 - (\bar{u}_1 - c_1)/(2t)]$, then retailer 1's profit π_1^3 is increasing in S_{11} and the retailer should thus determine his optimal host space as $S_{11}^* = S_1$. However, we find from Theorems 3 and 5 and Table 3 that the retailer's optimal price in retailer 2's store is $p_{12}^* = (\bar{u}_1 + c_1)/2$ and the resulting sales in his guest space is $B_1(\bar{u}_1 - c_1)/(2t)$, which is smaller than $B_1 - S_1$. Thus, when $S_1 < B_1[1 - (\bar{u}_1 - c_1)/(2t)]$, the retailer only needs the guest space $B_1(\bar{u}_1 - c_1)/(2t)$ and does not serve all consumers.

If $S_1 + S_{21} \geq B_1$ but $S_{21} < B_1$, then Table 3 indicates that retailer 1's maximum profit depends on his guest space in retailer 2's store. Similar to our above discussion when $S_1 < B_1/2$, we find that, if $B_1[1 - (\bar{u}_1 - c_1)/(2t)] \leq S_{21} < B_1/2$, then retailer 1 should accept the space S_{21} given by retailer 2, and he should determine his optimal host space as $S_{11}^* = B_1 - S_{21}$ and allocate the space $S_{12}^* = S_1 - B_1 + S_{21}$ to retailer 2. As a result, the retailer serves all of B_1 product 1 consumers. If $S_{21} < B_1[1 - (\bar{u}_1 - c_1)/(2t)]$, then retailer 1 still accepts the guest space S_{21} , but determines his optimal host space as $S_{11}^* = B_1(\bar{u}_1 - c_1)/(2t)$ and allocates to retailer 2 the space $S_{12}^* = S_1 - B_1(\bar{u}_1 - c_1)/(2t)$.

2. If $S_1 + S_{21} < B_1$, then retailer 1 cannot serve all of B_1 product 1 consumers. We learn from the proof of Lemma 3 that retailer 1's optimal host space should be $S_{11}^* = \min[B_1(\bar{u}_1 - c_1)/(2t), S_1]$. The retailer should allocate $S_{12}^* = S_1 - S_{11}^* = S_1 - \min[B_1(\bar{u}_1 - c_1)/(2t), S_1]$. Similarly, retailer 1 only accepts the space $\min[B_1(\bar{u}_1 - c_1)/(2t), S_{21}]$ from retailer 2.

Summarizing the above and similarly analyzing the best response for retailer 2, we have the lemma. ■

Proof of Lemma 7. The proof is the same as that for the case of $S_i + S_{ji} < B_i$ ($i, j = 1, 2$ and $i \neq j$) in Lemma 6. ■

Proof of Lemma 8. Since $S_i > \max[B_i[1 - (\bar{u}_i - c_i)/(2t)], B_i(\bar{u}_i - c_i)/(2t)]$, for $i = 1, 2$, we find from Lemma 7 that retailer i 's best-response space decision is $B_i(\bar{u}_i - c_i)/(2t)$. When two retailers retain their host shelf space in Nash equilibrium, we can find that, to implement the space-exchange strategy, retailer i allocates the space $S_{ij}^N = S_i - S_{ii}^N = S_i - B_i(\bar{u}_i - c_i)/(2t)$ to retailer j ($j = 1, 2$ and $j \neq i$), who allocates the space $S_{ji}^N = S_j - S_{jj}^N = S_j - B_j(\bar{u}_j - c_j)/(2t)$ to retailer i . Note that $S_{ij}^N > 0$ and $S_{ji}^N > 0$ because of the condition in Proposition 3, under which two retailers should decide to exchange shelf space. Then, using Lemma 5, we can compute two retailers' corresponding optimal prices as given in the lemma. ■

Proof of Lemma 9. To facilitate our proof, we consider the case that $\bar{u}_1 > c_1 + t$ and $\bar{u}_2 \leq c_2 + t$, and find the corresponding Nash equilibrium. According to Lemma 7, we find that, if $\bar{u}_2 \leq c_2 + t$, then retailer 2's optimal space decision S_{22}^* is the same as that when $S_2 + S_{12} < B_2$ in Table 4. That is, for this case, retailer 2's optimal space decisions is always $S_{22}^* = B_2(\bar{u}_2 - c_2)/(2tr_2)$; see Figure 4(a). Using our best-response analysis, we draw Figure 4(b) to show retailer 1's best space decision S_{11}^* given retailer 2's decision S_{22} .

Since the line $S_{22} = B_2(\bar{u}_2 - c_2)/(2tr_2)$ in Figure 4(a) may intersect with each of three line segments in Figure 4(b), there are three possible unique Nash equilibria, as given in Table 5. Replacing 1 and 2 with i and j ($i, j = 1, 2$ and $i \neq j$), we prove this lemma. ■

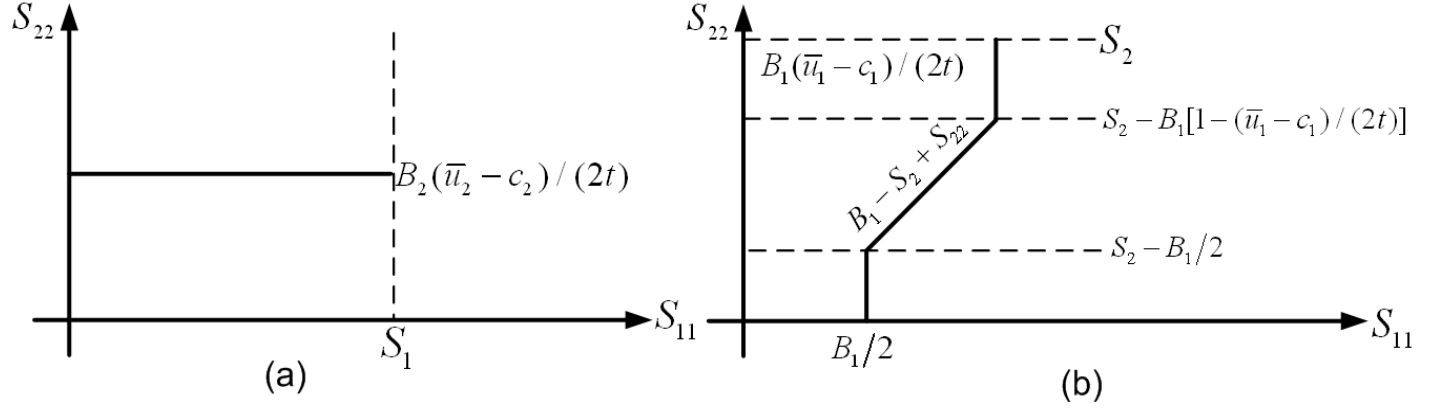


Figure 4: Two retailers' best-response space decisions when $\bar{u}_1 > c_1 + t$ and $\bar{u}_2 \leq c_2 + t$. Figure (a) and (b) indicate retailer 2's and retailer 1's best responses, respectively.

Proof of Lemma 10. We draw Figure 5 to show two retailers' best responses. To find Nash equilibrium, we need to discuss where two retailers' best-response functions intersect, because the intersection point represents a Nash equilibrium.

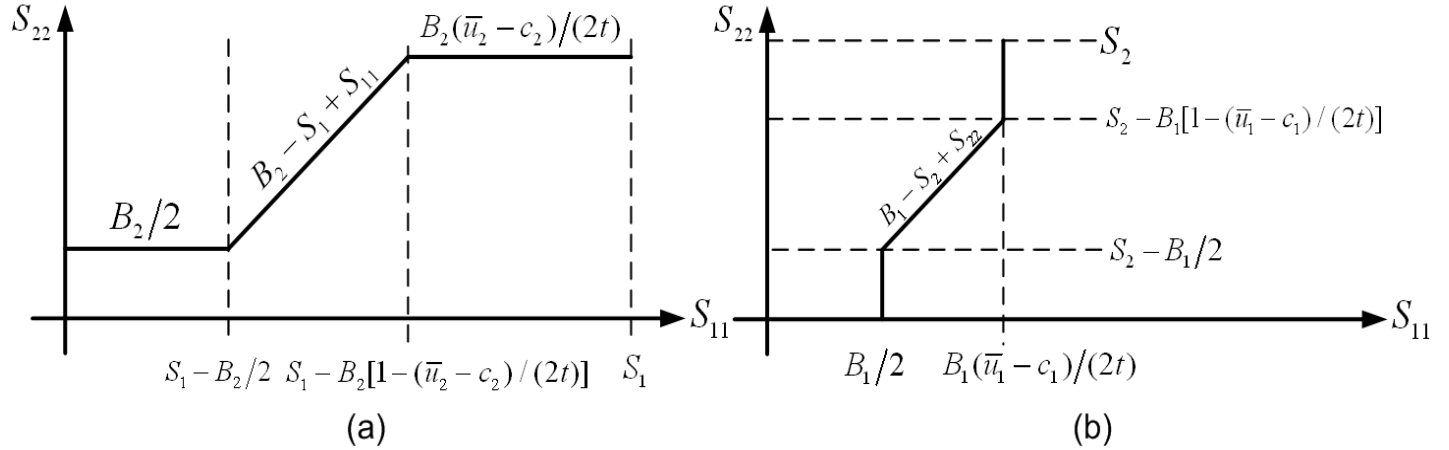


Figure 5: Two retailers' best-response space decisions when $\bar{u}_1 > c_1 + t$ and $\bar{u}_2 > c_2 + t$. Specifically, Figure (a) indicates retailer 2's best space decision S_{22} given retailer 1's decision S_{11} , and Figure (b) shows retailer 1's best space decision S_{11} given retailer 2's decision S_{22} .

As Figure 5 indicates, each retailer's best response is a step function consisting of three segments. Therefore, we have to consider 13 scenarios, as shown in Figure 6. For each scenario, we can calculate the corresponding Nash equilibrium, which is represented by the intersection of two retailers' best-response functions in Figure 6. For our solution, see Table 7 in online Appendix F. This proves the lemma. ■

Appendix B Proofs of Propositions

Proof of Proposition 1. This proposition follows Lemmas 4 and 5. That is, for both the case that $\bar{u}_i > c_i + t$ and the case that $\bar{u}_i \leq c_i + t$, if two retailers cooperate under the

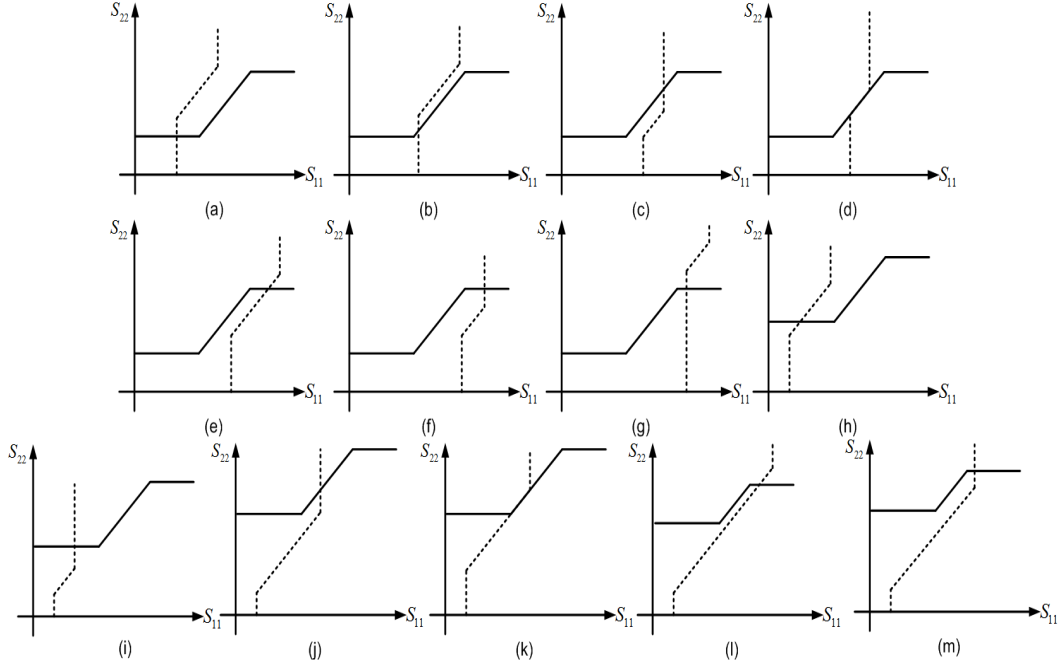


Figure 6: Nash equilibria in 13 scenarios. Note that the solid and the dashed step functions represent retailer 2's and retailer 1's best responses, respectively.

space-exchange strategy, retailer i 's prices in two stores (i.e., p_{i1}^* and p_{i2}^*) must be higher than the retailer's price p_i^* when two retailers do not exchange shelf space. ■

Proof of Proposition 2. From Lemmas 3 and 5, we learn that, if two retailers exchange shelf space, then each retailer may charge different prices in two stores. For example, as Lemma 3 indicates, $p_{ii}^* = \bar{u}_i - tS_{ii}/B_i$ is different from $p_{ij}^* = \max[(\bar{u}_i + c_i)/2, \bar{u}_i - t(1 - S_{ii}/B_i)]$, when $\bar{u}_i > c_i + t$, $S_{ii} + S_{ji} \geq B_i$, $S_{ii} < B_i/2$, and $S_{ji} > B_i/2$. This proposition is thus proved. ■

Proof of Proposition 3. The space-exchange strategy is implemented in the “simultaneous-move” game *if and only if* each retailer decides to allocate non-zero shelf space to the other retailer in Nash equilibrium. According to our best-response analysis, we find that, for the “simultaneous-move” game, two retailers should not exchange shelf space if and only if one or more of the following six things happen: (i) $S_i \leq B_i/2$; (ii) $S_i \leq B_i[1 - (\bar{u}_i - c_i)/(2t)]$; (iii) $S_i \leq B_i(\bar{u}_i - c_i)/(2t)$; (iv) $S_j \leq B_j/2$; (v) $S_j \leq B_j[1 - (\bar{u}_j - c_j)/(2t)]$; and (vi) $S_j \leq B_j(\bar{u}_j - c_j)/(2t)$. Note that

$$\max[B_i[1 - (\bar{u}_i - c_i)/(2t)], B_i(\bar{u}_i - c_i)/(2t)] > B_i/2, \text{ for } i = 1, 2.$$

It thus follows that, in Nash equilibrium, two retailers decide to exchange shelf space if and only if $S_i > \max[B_i[1 - (\bar{u}_i - c_i)/(2t)], B_i(\bar{u}_i - c_i)/(2t)]$. ■

Appendix C Detailed Discussion for Remark 1

We find from (3) that a product i ($i = 1, 2$) consumer has no difference between buying in the two stores if $\hat{u}_{x1} = \hat{u}_{x2}$, or, $\hat{x}_i \equiv (p_{i2} - p_{i1} + t)/2t$. That is, the product i consumer at location \hat{x}_i can obtain the same utility when he or she buys in retailer 1's or retailer 2's store.

As discussed in Section 2, because the B_i product i consumers are uniformly distributed along the linear city between the two retailers' stores (i.e., between the end points 0 and 1), retailer i should make his pricing decision (p_{i1}, p_{i2}) such that $0 \leq \hat{x}_i \leq 1$, or, $-t \leq p_{i2} - p_{i1} \leq t$. In addition, whether or not any consumer residing between the two retailers' stores buys from a store depends on the condition that $\hat{u}_{xi} \geq 0$. According to (3), we find that, given the retail price p_{i1} , any consumer who is located at $x \leq \hat{x}_{i1} \equiv (\bar{u}_i - p_{i1})/t$ should visit retailer 1's store at the end point 0 to buy a unit of product i . In order to effectively serve product i consumers, retailer i should make his pricing decision p_{i1} such that $\hat{x}_{i1} \geq 0$, or, $p_{i1} \leq \bar{u}_i$. Similarly, given the retail price p_{i2} , any consumer at the location $x \geq \hat{x}_{i2} \equiv (p_{i2} + t - \bar{u}_i)/t$ should buy product i from retailer i at retailer 2's store. Retailer i 's price p_{i2} for product i in retailer 2's store should be smaller than or equal to \bar{u}_i —i.e., $p_{i2} \leq \bar{u}_i$ —so as to assure that $\hat{x}_{i2} \leq 1$.

Noting that $\hat{x}_{i1} + \hat{x}_{i2} \equiv 2\hat{x}_i$, we find that, as Figure 7 indicates, either of the two following cases happens: (i) $\hat{x}_{i1} \geq \hat{x}_i$ and $\hat{x}_{i2} \leq \hat{x}_i$; or, (ii) $\hat{x}_{i1} \leq \hat{x}_i$ and $\hat{x}_{i2} \geq \hat{x}_i$. Accordingly, we consider the following two cases to compute the demands faced by retailer i in two retailers' stores.

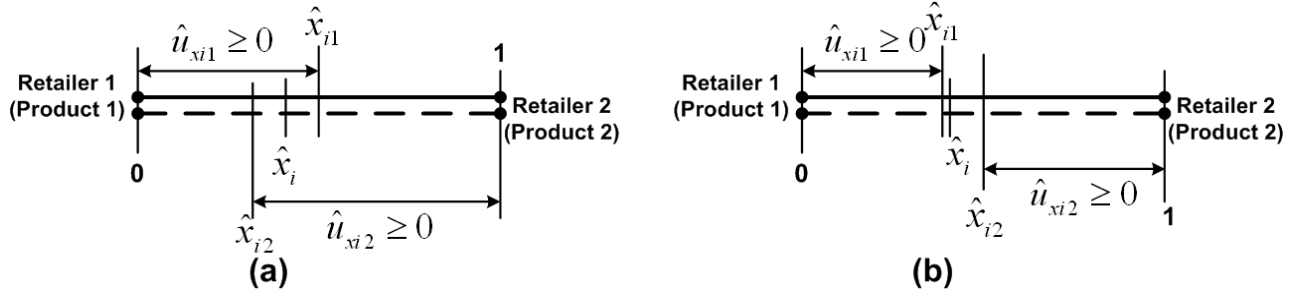


Figure 7: The demands faced by retailer i ($i = 1, 2$) in two retailers' stores.

1. If $\hat{x}_{i1} \geq \hat{x}_i$, or, $p_{i1} + p_{i2} \leq 2\bar{u}_i - t$, then $\hat{x}_{i2} \leq \hat{x}_i$. We learn from Figure 7(a) that the product i consumers between the point 0 and the point \hat{x}_{i1} can gain a non-negative utility from buying in retailer 1's store, and those consumers between the point \hat{x}_{i2} and the point 1 have a non-negative utility from buying in retailer 2's store. For this case, all of B_i product i consumers' net utilities must be non-negative when they buy in either retailer 1's or retailer 2's store, and these consumers should be thus willing to complete transactions with retailer 1. We also find that any consumer residing between the points \hat{x}_{i2} and \hat{x}_{i1} can draw a non-negative utility from purchasing from both retailer 1's and retailer 2's stores. Such a consumer should choose a store where his or her net utility is higher. Therefore, as Figure 7(a) indicates, any product i consumer residing between the end point 0 and the point \hat{x}_i should buy from retailer 1's store, and any product 1 consumer between \hat{x}_i and the end point 1 should buy from retailer 2's store. Therefore, the demands faced by retailer i in the two stores are computed as,

$$D_{i1} = B_i \frac{p_{i2} - p_{i1} + t}{2t} \quad \text{and} \quad D_{i2} = B_i \frac{p_{i1} - p_{i2} + t}{2t}.$$

It thus follows that, if retailer i makes his pricing decision (p_{i1}, p_{i2}) such that $p_{i1} + p_{i2} \leq 2\bar{u}_i - t$, then retailer i should serve all consumers in the market.

2. If $\hat{x}_{i2} \geq \hat{x}_i$, or, $p_{i1} + p_{i2} \geq 2\bar{u}_i - t$, then $\hat{x}_{i1} \leq \hat{x}_i$. Similar to the first case, we note from Figure 7(b) that any consumer residing between the points \hat{x}_{i1} and \hat{x}_{i2} cannot draw a non-negative utility from shopping in each store, and should be thus unwilling to buy from

retailer 1. This means that, for this case, some consumer(s) may not decide to purchase product i . Figure 7(b) indicates that only consumers between the points 0 and \hat{x}_{i1} should buy in retailer 1's store, and only consumers between \hat{x}_{i2} and 1 should buy in retailer 2's store. The demands faced by retailer i in the two stores are thus computed as,

$$D_{i1} = B_i \frac{\bar{u}_i - p_{i1}}{t} \quad \text{and} \quad D_{i2} = B_i \frac{\bar{u}_i - p_{i2}}{t}.$$

The total demand faced by retailer i is thus $B_i(2\bar{u}_i - p_{i1} - p_{i2})/t$; this means that, if retailer i 's prices (p_{i1}, p_{i2}) are determined such that $p_{i1} + p_{i2} \geq 2\bar{u}_i - t$, then he shall only serve $(2\bar{u}_i - p_{i1} - p_{i2})/t$ (rather than all) of B_i consumers in the market.

From the above we find that all product i consumers buy when $\hat{x}_{i1} \geq \hat{x}_i$ whereas some consumer(s) may not buy when $\hat{x}_{i2} \geq \hat{x}_i$. Note that, if $p_{i1} + p_{i2} = 2\bar{u}_i - t$, then $\hat{x}_{i1} = \hat{x}_{i2} = \hat{x}_i$, and the demands for the above two cases are the same, which means that all consumers will buy but no consumer can enjoy a non-negative utility from buying from both stores.

Appendix D Proof of Corollary 1

We first compare the profits for retailer 1. For this corollary, we should compare the profit given in Table 3 and that given in Lemma 1. As Lemma 1 indicates, retailer 1's maximum profit when two retailers do not exchange shelf space is dependent on the value of the total space S_1 in retailer 1's store; but, we find from Table 3 that, when two retailers implement the space-exchange strategy, retailer 1's maximum profit depends on the values of S_{11} and S_{21} . This means that, for each profit in Lemma 1, we have to compare it with any profit in Table 3. For example, if $S_1 \geq B_1$, then Lemma 1 indicates that retailer 1's maximum profit with no space exchange is either $B_1 \times (\bar{u}_1 - c_1)^2/(4t)$ (when $c_1 + 2t \geq \bar{u}_1$) or $B_1 \times (\bar{u}_1 - t - c_1)$ (when $c_1 + 2t \leq \bar{u}_1$). For this case, under the space-exchange strategy, the retailer's maximum profit could be any value of π_1^i ($i = 1, \dots, 8$).

1. If $S_1 \geq B_1$ and $c_1 + t < \bar{u}_1 \leq c_1 + 2t$, then retailer 1's maximum profit with no space exchange is $\pi_1^* = B_1 \times (\bar{u}_1 - c_1)^2/(4t)$. It is easy to find that $\pi_1^3, \pi_1^5, \pi_1^7$, and π_1^8 are all greater than π_1^* . Next, we need to consider $\pi_1^1, \pi_1^2, \pi_1^4$, and π_1^6 .

We first compare π_1^1, π_1^2 , and π_1^3 . It is easy to note that $\pi_1^2 < B_1(\bar{u}_1 - t - c_1) + B_1t(1 - S_{11}/B_1)$ because $S_{11} < B_1/2$. Furthermore, we have,

$$B_1(\bar{u}_1 - t - c_1) + B_1t(1 - S_{11}/B_1) = B_1(\bar{u}_1 - tS_{11}/B_1 - c_1) < B_1(\bar{u}_1 - t/2 - c_1) = \pi_1^1,$$

which means that $\pi_1^2 < \pi_1^1$. In fact, we can easily show that, when $S_{11} = B_1/2$, π_1^2 arrives to its maximum value that is equal to π_1^1 . In addition, we find that π_1^3 is maximized when $S_{11}/B_1 = (\bar{u}_1 - c_1)/(2t)$. However, the retailer cannot choose the space S_{11} such that $S_{11}/B_1 = (\bar{u}_1 - c_1)/(2t)$, because of the following fact: If $\bar{u}_1 - c_1 > 2t$, then $\bar{u}_1 > c_1 + 2t(1 - S_{11}/B_1)$ and, as Table 3 indicates, retailer 1's profit is π_1^2 rather than π_1^3 . If $\bar{u}_1 - c_1 \leq 2t$, then retailer 1's profit is π_1^3 only when $\bar{u}_1 \leq c_1 + 2t(1 - S_{11}/B_1)$, or, $S_{11}/B_1 \leq 1 - (\bar{u}_1 - c_1)/(2t)$. Noting that $1 - (\bar{u}_1 - c_1)/(2t) < 1/2$ and $(\bar{u}_1 - c_1)/(2t) > 1/2$ because $\bar{u}_1 - c_1 > t$, we find that the profit π_1^3 is increasing in S_{11} when $S_{11} \in [0, B_1(1 - (\bar{u}_1 - c_1)/(2t))]$, and it thus follows that when $S_{11} = B_1[1 - (\bar{u}_1 - c_1)/(2t)]$, π_1^3 arrives to its maximum π_1^2 , which is smaller than π_1^1 , as argued above.

- Since $\pi_1^1 > \pi_1^2 \geq \pi_1^3$, we find that both π_1^1 and π_1^2 are greater than π_1^* . Similarly, we can also show that $\pi_1^4 \geq \pi_1^5$; thus, π_1^4 is greater than π_1^* . However, π_1^6 may or may not be greater than π_1^* . Noting that $\pi_1^6 \geq (S_{11} + S_{21})(\bar{u}_1 - c_1)/2$, we can derive a sufficient condition under which $\pi_1^6 \geq \pi_1^*$ as $S_{11} + S_{21} \geq B_1(\bar{u}_1 - c_1)/(2t)$, or, $S_{11} + S_{21} \geq B_1(\bar{u}_1 - c_1)/(2t)$.
2. If $S_1 \geq B_1$ and $\bar{u}_1 \geq c_1 + 2t$, then retailer 1's maximum profit with no space exchange is $\pi_1^* = B_1 \times (\bar{u}_1 - t - c_1)$. For this case, under the space-exchange strategy, the retailer's profit should be one of $\pi_1^1, \pi_1^2, \pi_1^4$, and π_1^6 . It is easy to find that π_1^1, π_1^2 , and π_1^4 are all greater than π_1^* . But, π_1^6 may or may not be greater than π_1^* .
 3. If $S_1 < B_1$ and $c_1 + 2tS_1/B_1 \geq \bar{u}_1$, then retailer 1's profit with no space exchange $\pi_1^* = B_1 \times (\bar{u}_1 - c_1)^2/(4t)$. This is the same as that when $S_1 \geq B_1$ and $c_1 + t < \bar{u}_1 \leq c_1 + 2t$.
 4. If $S_1 < B_1$ and $c_1 + 2tS_1/B_1 \leq \bar{u}_1$, then $\pi_1^* = (\bar{u}_1 - tS_1/B_1 - c_1) \times S_1$, which is smaller than or equal to $B_1 \times (\bar{u}_1 - c_1)^2/(4t)$ because we can easily show that, for this case, π_1^* arrives to its maximum value $B_1 \times (\bar{u}_1 - c_1)^2/(4t)$ when $c_1 + 2tS_1/B_1 = \bar{u}_1$. Therefore, we can find that π_1^6 may or may not be greater than π_1^* and π_1^i ($i = 1, \dots, 5, 7, \dots, 8$) is greater than π_1^* .

The analysis for retailer 2 is similar to the above. This corollary is thus proved.

Appendix E A Further Discussion on the Result in Proposition 3

We learn from Proposition 3 that each retailer's total shelf space should be sufficiently large in order to assure that two retailers are willing to implement the space-exchange strategy in the non-cooperative game. One may note that two retailers with small shelf space could be also likely to consider the space-exchange strategy. For example, suppose that both retailers 1 and 2 have very small shelf space; e.g., retailer i ($i = 1, 2$) can stock only two units of product i in his store before the space exchange, i.e., $S_i = 2$, for $i = 1, 2$. When two retailer do not exchange shelf space, retailer i would set his price such that the *two* consumers who are the closest to the retailer along the Hotelling line would find it worthwhile to buy product i . If two retailers exchange shelf space, then they may raise their prices without losing any consumers, and their profits could thus be higher than those in the "no space exchange" case, because of the following reason: When retailer i allocates a unit of shelf space in his own store to retailer j ($j = 1, 2, j \neq i$), the retailer's host shelf space can be used to serve only the customer who is the closed to retailer i 's store. That is, $S_{ii} = S_{ij} = 1$. Hence, in retailer i 's own store, the retailer does not need to consider the second closest customer, and can increase his price to a value such that only the closest customer is willing to buy. Meanwhile, retailer i obtains a unit of guest shelf space from retailer j , and uses it to serve the product i customer who is the closest to retailer j 's store, setting his retail price as in his own store. As a result of exchanging shelf space with retailer j , retailer i 's prices (in both stores) are higher than his price before the space exchange, and the retailer can still serve two customers, as in the "no space exchange" case.

The above discussion may deliver a result different from that in Proposition 3. In fact, such a difference appears mainly because Proposition 3 holds when two retailers make their decisions in the *non-cooperative* game whereas the above discussion is based on the assumption that two retailers jointly make their decisions in the *cooperative* setting. We re-consider the above example where each retailer's shelf space can be used to serve only two customers. When retailer

j allocates a unit of shelf space to retailer i , retailer i 's best response in the non-cooperative setting is not to also allocate a unit of shelf space to retailer j but to keep all of his shelf space S_i , because, as discussed in Section 4.1, retailer i 's profit is increasing in the space S_i when S_i is so small that retailer i cannot serve a half of B_i consumers [i.e., $S_i \leq B_i(\bar{u}_i - c_i)/(2t)$]. Hence, two retailers' space allocation decisions ($S_{ii} = S_{jj} = 1$, for $i, j = 1, 2$ and $i \neq j$) cannot be obtained in equilibrium. That is, two retailers' decisions in Nash equilibrium may not be the Pareto optimal. Such a result can be actually regarded as "Prisoner's Dilemma," which is indicated as in Table 6 where we note that there is a unique Nash equilibrium $(S_{11}, S_{22}) = (2, 2)$ because $\pi_1(2, 1) > \pi_1(1, 1) > \pi_1(2, 0) > \pi_1(1, 0)$ and $\pi_2(1, 2) > \pi_2(1, 1) > \pi_2(0, 2) > \pi_2(0, 1)$. For more information regarding Prisoner's Dilemma, see, e.g., Straffin [13, Ch. 12].

		Retailer 1	
		Allocate	Do Not Allocate
Retailer 2	Allocate	$(\pi_1(1, 1), \pi_2(1, 1))$ [$(S_{11}, S_{22}) = (1, 1)$]	$(\pi_1(2, 1), \pi_2(0, 1))$ [$(S_{11}, S_{22}) = (2, 1)$]
	Do Not Allocate	$(\pi_1(1, 0), \pi_2(1, 2))$ [$(S_{11}, S_{22}) = (1, 2)$]	$(\pi_1(2, 0), \pi_2(0, 2))$ [$(S_{11}, S_{22}) = (2, 2)$]

Table 6: The "Prisoner's dilemma" in the space-exchange game. Retailer i ($i = 1, 2$)'s decision "Allocate" means that the retailer decides to allocate a unit of his shelf space to retailer j ($j = 1, 2, j \neq i$), and Retailer i 's decision "Do Not Allocate" means that the retailer does not allocate any space to retailer j but keeps all of his space S_i . In addition, $\pi_i(S_{ii}, S_{ji})$ and $\pi_j(S_{ij}, S_{jj})$ denote retailers i 's and j 's profits when the retailers' space decisions are S_{ii} and S_{jj} , respectively. Note that $\pi_1(2, 1) > \pi_1(1, 1) > \pi_1(2, 0) > \pi_1(1, 0)$ and $\pi_2(1, 2) > \pi_2(1, 1) > \pi_2(0, 2) > \pi_2(0, 1)$.

Moreover, if a retailer's store is very small, then two retailers may be willing to open new stores instead of exchange shelf space (if their costs of opening and staffing new stores are not large). For a specific discussion on the impact of fixed costs, see Section 5.2. Our result in Proposition 3 is also in gear with the practice that the retailers exchanging shelf space include, e.g., Waitrose, Boots, Tim Hortons, and Cold Stone Creamery.

Appendix F Nash Equilibrium when $\bar{u}_i > c_1 + t$ ($i = 1, 2$)

For our game analysis for the case that $\bar{u}_i > c_i + t$ ($i = 1, 2$) in Section 4.2, we provide thirteen possible Nash equilibria in Table 7.

Scenario	Conditions	Nash Equilibrium
(a)	$S_1 \geq \frac{B_1}{2} + \frac{B_2}{2}, S_2 \geq \frac{B_1}{2} + \frac{B_2}{2}$	$S_{11}^N = \frac{B_1}{2}, S_{22}^N = \frac{B_2}{2}$
(b)	$B_1 + B_2 - S_2 \leq S_1 \leq \frac{B_1}{2} + \frac{B_2}{2},$ $S_2 \geq \frac{B_1}{2} + \frac{B_2}{2};$ or $\frac{B_1}{2} + B_2[1 - (\bar{u}_2 - c_2)/(2t)] \leq S_1 \leq \frac{B_1}{2} + \frac{B_2}{2},$ $S_2 \geq \frac{B_1}{2} + \frac{B_2(\bar{u}_2 - c_2)}{2t}$	$S_{11}^N = \frac{B_1}{2},$ $S_{22}^N = B_1 + B_2 - S_1$
(c)	$\frac{B_1(\bar{u}_1 - c_1)}{2t} - \frac{B_2(\bar{u}_2 - c_2)}{2t} + B_2 \leq S_1 \leq B_1 + B_2 - S_2,$ $S_2 \geq \frac{B_1}{2} + \frac{B_2}{2}$	$S_{11}^N = \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_{22}^N = B_2 + \frac{B_1(\bar{u}_1 - c_1)}{2t} - S_1$
(d)	$S_1 = B_1 + B_2 - S_2,$ $S_2 \geq \frac{B_1}{2} + \frac{B_2}{2}$	$\frac{B_1}{2} \leq S_{11}^N \leq \min \left[\frac{B_1(\bar{u}_1 - c_1)}{2t}, S_1 - B_2[1 - (\bar{u}_2 - c_2)/(2t)] \right],$ $S_{22}^N = B_2 - S_1 + S_{11}^N$
(e)	$S_1 < B_1 + B_2 - S_2,$ $S_2 \geq B_1[1 - (\bar{u}_1 - c_1)/(2t)] + \frac{B_2(\bar{u}_2 - c_2)}{2t}$	$S_{11}^N = B_1 + \frac{B_2(\bar{u}_2 - c_2)}{2t} - S_2,$ $S_{22}^N = \frac{B_2(\bar{u}_2 - c_2)}{2t}$
(f)	$S_1 < B_1 + B_2 - S_2,$ $S_2 \leq B_1[1 - (\bar{u}_1 - c_1)/(2t)] + \frac{B_2(\bar{u}_2 - c_2)}{2t}$	$S_{11}^N = \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_{22}^N = \frac{B_2(\bar{u}_2 - c_2)}{2t}$
(g)	$S_1 \leq \frac{B_1}{2} + B_2[1 - (\bar{u}_2 - c_2)/(2t)], S_2 \geq \frac{B_1}{2} + \frac{B_2(\bar{u}_2 - c_2)}{2t}$	$S_{11}^N = \frac{B_1}{2}, S_{22}^N = \frac{B_2(\bar{u}_2 - c_2)}{2t}$
(h)	$S_1 \geq B_1 + B_2 - S_2,$ $\frac{B_2}{2} + B_1[1 - (\bar{u}_1 - c_1)/(2t)] \leq S_2 \leq \frac{B_1}{2} + \frac{B_2}{2}$	$S_{11}^N = B_1 + B_2 - S_2,$ $S_{22}^N = \frac{B_2}{2}$
(i)	$S_1 \geq \frac{B_2}{2} + \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_2 \leq \frac{B_2}{2} + B_1[1 - (\bar{u}_1 - c_1)/(2t)]$	$S_{11}^N = \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_{22}^N = \frac{B_2}{2}$
(j)	$\frac{B_1(\bar{u}_1 - c_1)}{2t} + B_2[1 - (\bar{u}_2 - c_2)/(2t)] \leq S_1 \leq \frac{B_2}{2} + \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_2 < B_1 + B_2 - S_1$	$S_{11}^N = \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_{22}^N = \frac{B_2}{2} + \frac{B_1(\bar{u}_1 - c_1)}{2t} - S_1$
(k)	$\frac{B_1(\bar{u}_1 - c_1)}{2t} + B_2[1 - (\bar{u}_2 - c_2)/(2t)] \leq S_1 \leq \frac{B_2}{2} + \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_2 = B_1 + B_2 - S_1$	$S_1 - \frac{B_2}{2} \leq S_{11}^N \leq \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_{22}^N = B_2 - S_1 + S_{11}^N$
(l)	$S_1 \leq \frac{B_1(\bar{u}_1 - c_1)}{2t} + B_2[1 - (\bar{u}_2 - c_2)/(2t)],$ $S_2 \geq \max \left[\frac{B_2(\bar{u}_2 - c_2)}{2t} + B_1[1 - (\bar{u}_1 - c_1)/(2t)], B_1 - B_2 + S_1 \right]$	$S_{11}^N = \frac{B_2(\bar{u}_2 - c_2)}{2t} + \frac{B_1}{2} - S_2,$ $S_{22}^N = \frac{B_2(\bar{u}_2 - c_2)}{2t}$
(m)	$S_1 \leq \frac{B_1(\bar{u}_1 - c_1)}{2t} + B_2[1 - (\bar{u}_2 - c_2)/(2t)],$ $S_2 \leq \frac{B_2(\bar{u}_2 - c_2)}{2t} + B_1[1 - (\bar{u}_1 - c_1)/(2t)]$	$S_{11}^N = \frac{B_1(\bar{u}_1 - c_1)}{2t},$ $S_{22}^N = \frac{B_2(\bar{u}_2 - c_2)}{2t}$

Table 7: Nash equilibria for 13 scenarios when $\bar{u}_1 > c_1 + t$ and $\bar{u}_2 > c_2 + t$.