Multi-Player Allocations in the Presence of Diminishing Marginal Contributions: Cooperative Game Analysis and Applications in Management Science

Mingming Leng[1], Chunlin Luo[2], Liping Liang[3]

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Abstract

We use cooperative game theory to investigate multi-player allocation problems under the almost diminishing marginal contributions (ADMC) property. This property indicates that a player's marginal contribution to a non-empty coalition decreases as the size of the coalition increases. We develop ADMC games for such problems and derive a necessary and sufficient condition for the non-emptiness of the core. When the core is non-empty, at least one extreme point exists, and the maximum number of extreme points is the total number of players. The Shapley value may not be in the core, which depends on the gap of each coalition. A player can receive a higher allocation based on the Shapley value in the core than based on the nucleolus, if the gap of the player is no greater than the gap of the complementary coalition. We also investigate the least core value for ADMC games with an empty core. To illustrate the applications of our results, we analyze a code-sharing game, a group-buying game, and a scheduling profit game.

Keywords: Coalitional games; diminishing contributions; the core; the nucleolus; the Shapley value.

1 Introduction

Cooperative game theory is concerned with the allocation of a payoff that is jointly made by players in a multi-player coalition. Since the 1940s, a number of solution concepts have been proposed to solve the problem of how to allocate the coalition payoff in a fair manner. Among these solution concepts, the core (Gillies 1953), the Shapley value (Shapley 1953), and the nucleolus (Schmeidler 1969) have received the most attention from researchers in the management science area. Recent decades have witnessed a proliferation of publications that use the theory in the analysis of allocation-related management science problems, including those of Hartman and Dror (1996), Raghunathan (2003), Sošić (2006), Dror and Hartman (2007), Nagarajan and Sošić (2007, 2009), Nagarajan and Bassok (2008), and Leng and Parlar (2009).

The core (Gillies 1953) usually does not represent a unique allocation scheme. To test the non-emptiness of the core of a coalitional game, we commonly use the Bondareva-Shapley theorem (Bondareva 1963 and Shapley 1967) to check the balancedness of the game. This theorem may require us

[1] Faculty of Business, Lingnan University, Hong Kong. (Email: mmleng@Ln.edu.hk; ORCID ID: https://orcid.org/0000-0002-3377-2387)

[2] School of Information Management, Jiangxi University of Finance and Economics, Nanchang 330013, China. (Email: chunlinluo@126.com; ORCID ID: https://orcid.org/0000-0002-0917-4264)

[3] Faculty of Business, Lingnan University, Hong Kong. (Email: lipingliang@ln.edu.hk; ORCID ID: https://orcid.org/0000-0002-6491-1503)

to examine a large number of proper minimal balanced sets, especially when the number of players is large (Shapley 1967). Although the Shapley value (Shapley 1953) can be computed with a formula, it does not always belong to the non-empty core. Thus, the Shapley value may not ensure the stability of the grand coalition. In addition, because the Shapley value requires an evaluation for each subset, its computation is exponentially difficult with a large number of players. The nucleolus (Schmeidler 1969) is an attractive solution concept, mainly because it always uniquely exists in the non-empty core. Nonetheless, it may not be easy to obtain the nucleolus because we usually have to calculate it numerically by solving a series of linear programming (LP) problems in an iterative manner or by solving a large-scale LP problem. Leng and Parlar (2010) provided a summary regarding the complexity of computing the nucleolus. The above implies that some properties of the common solution concepts may not be desirable to researchers and practitioners.

The undesirable properties mentioned above could limit the applications of the core, the Shapley value, and the nucleolus in the management science area. It behooves us to address the following critical question: does a class of n-player ($n \geq 3$) coalitional games possess the properties that allow us to (i) derive a closed-form condition to examine the non-emptiness of the core, (ii) obtain a closed-form nucleolus solution, and (iii) find a closed-form condition to test whether the Shapley value is in the core? This question is important because we often need closed-form conditions and expressions to analyze allocation-related problems in management science. To that end, in this paper we find a class of coalitional games called "almost diminishing marginal contributions (ADMC) games." This class of games is developed according to the well-known "law of diminishing returns," that is, a decrease in the marginal output as a result of an incremental increase in an input. For details, see, e.g., Samuelson and Nordhaus (2009). An n-player ADMC game is a coalitional game that satisfies the ADMC property, that is, a player's marginal contribution to a non-empty coalition decreases as the size of the coalition increases.

We derive a necessary and sufficient condition to check the balancedness of an ADMC game. We also find that the closed-form nucleolus solution for an ADMC game with a non-empty core is identical to the Equal Allocation of Non-Separable Contribution/Cost (EANSC) value (Moulin 1985). The non-separable contribution/cost (NSC) is computed as the value of the grand coalition minus the sum of all players' (separable) marginal contributions to the grand coalition. If the NSC is non-negative, it represents the non-separable contribution; otherwise, it means the non-separable cost. For an n-player ADMC game with a non-empty core, we obtain the extreme points of the core in closed form and show the existence of at least one extreme point and at most n extreme points.

Regarding the Shapley value for the class of ADMC games, we address the following three questions. First, is the Shapley value always in the core of an ADMC game with a non-empty core? Second, can we derive a closed-form condition for the existence of the Shapley value in the core? Third, which players can receive higher allocations based on the Shapley value in the core than based on the nucleolus and would thus prefer to use the Shapley value for payoff allocation? For the first and second questions, we show that the Shapley value may not be in the core and derive a sufficient condition that is dependent on the gap of each $(n-1)$-player coalition. Although the literature is silent on the discussion of the third question or any similar one, this question is important because the grand payoff to be allocated among all players is a fixed amount. If the Shapley value and the nucleolus differ, then some players receive higher allocations under the Shapley value–based scheme, whereas some others receive more from the nucleolus. It is thus interesting to examine each player's preferred allocation scheme. Our analysis reveals that a player prefers the Shapley value to the nucleolus if the gap of the player is no greater than the gap of the $(n-1)$-player complementary coalition (i.e., the grand coalition minus the player).

We also investigate ADMC games with an empty core, for which a common solution concept is the least core value (Shapley and Shubik 1966, Maschler, Peleg, and Shapley 1979). Due to the complexity of computing the least core value, we explore the existence of a closed-form solution of the least core value for an ADMC game with an empty core. We derive a lower bound for the least core value, and obtain the closed-form solution under a condition. The lower bound and the least core value are both obtained as the equal allocation of the NSC. We illustrate our result by solving a prize donation game (Shapley and Shubik 1969). Moreover, to demonstrate the applicability of ADMC games in management science, we analyze three games: a code-sharing game, a group-buying game, and a scheduling profit game.

2 Definition and Analysis of ADMC Games

We consider an n-player coalitional game in characteristic function form $\mathcal{G} \equiv (N, v)$, where $N \equiv \{1, 2, \ldots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\varnothing) = 0$. Any subset of players is called a coalition. For each coalition $S \subseteq N$, the characteristic value $v(S)$ stands for the payoff that can be jointly obtained by all players in coalition S.

Definition 1 *A coalitional game $\mathcal{G}$ is an almost diminishing marginal contributions (ADMC) game if $v(i) \leq v(T \cup \{i\}) - v(T) \leq v(S \cup \{i\}) - v(S)$, $\forall i \in N$, $\forall S \subseteq T \subseteq N \backslash \{i\}$, and $S \neq \varnothing$.*

The condition in Definition 1 indicates the ADMC property, which can be viewed as a description of the law of diminishing returns (a fundamental principle of economics, see Samuelson and Nordhaus 2009) with the concepts and terms in cooperative game theory. Three examples are provided in Section 3 to illustrate ADMC games.

2.1 Non-Emptiness of the Core and Extreme Points of a Non-Empty Core

For an ADMC game, we derive a necessary and sufficient condition for the non-emptiness of the core and investigate the extreme points of the non-empty core.

2.1.1 A Necessary and Sufficient Condition for Non-Emptiness of the Core

When the core of game $\mathcal{G}$ is non-empty, each player has an incentive to remain in the grand coalition N if we use an allocation scheme in the core to divide the grand payoff $v(N)$ among n players. For game $\mathcal{G}$, x_i $(i \in N)$ represents a payoff allocated to player i in the grand coalition, i.e., $v(N) = \sum_{i \in N} x_i$. We can compute the excess of coalition $S \subseteq N$ as $e_S(\mathbf{x}) = v(S) - \sum_{j \in S} x_j$, where $\mathbf{x} \equiv (x_1, x_2, \ldots, x_n)$

is an n-tuple imputation. Moreover, $e_S(\mathbf{x}) \le 0$, $\forall S \subseteq N$, if and only if game $\mathcal{G}$ has a non-empty core and the imputation $\mathbf{x}$ is in the core.

Lemma 1 *An ADMC game with a non-empty core possesses the following two properties.*

1. *At an imputation $\mathbf{x}$ in the core, the excess of any coalition S increases as one more player joins coalition S; that is, $e_S(\mathbf{x}) \le e_{S\cup\{i\}}(\mathbf{x})$, $\forall i \in N\backslash S$ and $1 \le |S| \le n-1$.*
2. *The largest excess reaches its minimum when the excesses of all $(n-1)$-player coalitions are equal, i.e., $e_I(\mathbf{x}) = e_J(\mathbf{x})$, $\forall I, J \subseteq N$, $I \ne J$, and $|I| = |J| = n-1$.*

Then, we can obtain a necessary and sufficient condition for the non-emptiness of the core.

Theorem 1 *For an ADMC game, the core is non-empty if and only if the characteristic function v satisfies the following condition:*

$$\textstyle\sum_{j\in N} v(N\backslash\{j\}) \le (n-1)v(N). \tag{1}$$

We rewrite the condition in (1) as

$$\textstyle\sum_{j\in N} \left(v(N) - v(N\backslash\{j\})\right) \ge v(N), \tag{2}$$

where the term $v(N) - v(N\backslash\{j\})$ ($j \in N$) is the marginal contribution of player j to the grand coalition, thus reflecting player j's "added value" to the grand coalition. If the sum of all players' marginal contributions to the grand coalition is no less than the grand payoff $v(N)$, then at least one allocation of the grand payoff exists among all players such that no player has an incentive to leave the grand coalition. In addition, using (2), we find that, for an ADMC game with a non-empty core, the NSC of all players (Moulin 1985) is non-positive, i.e.,

$$\xi \equiv v(N) - \textstyle\sum_{j\in N}(v(N) - v(N\backslash\{j\})) \le 0. \tag{3}$$

Example 1 We consider a three-player coalitional game $\mathcal{G}_1 = (N, v_1)$, where $N = \{1,2,3\}$ and the characteristic values of all possible coalitions are

$$v(\varnothing) = 0;\ v(1) = 0;\ v(2) = 1;\ v(3) = 2;$$
$$v(12) = 4;\ v(13) = 5;\ v(23) = 7;\ v(123) = 9.$$

It follows from Definition 1 that $\mathcal{G}_1$ is an ADMC game. Moreover, using Theorem 1, we conclude that game $\mathcal{G}_1$ has a non-empty core.

In another three-player coalitional game $\mathcal{G}_2 = (N, v_2)$, where $v(23) = 6$, $v(123) = 7$, and the characteristic values of other coalitions are the same as those in coalitional game $\mathcal{G}_1$. We find from Definition 1 and Theorem 1 that $\mathcal{G}_2$ is an ADMC game with an empty core. $\triangleleft$

2.1.2 Extreme Points of a Non-Empty Core

In cooperative game theory, an important concept related to a non-empty core is the extreme point of the core. In the point set topology, an extreme point is a point in a convex set that does not lie on the open line segment connecting any two points in the set. That is, an extreme point of a convex set A is a point $\kappa \in A$ such that, if $\kappa = \theta\omega + (1-\theta)\tau$ with ω, $\tau \in A$ and $\theta \in [0,1]$, then $\omega = \kappa$ and/or $\tau = \kappa$. For any n-player coalitional game with a non-empty core, the maximum number of extreme points of the core is $n!$.

Because the non-empty core of a coalitional game is defined by a number of linear inequalities, it includes the interior and the boundary of a polytope and is thus convex. The extreme points of the core are the intersections of the planar facets of this polytope. Thus, an extreme point of the core (denoted by $\boldsymbol{\rho} \equiv (\rho_1, \rho_2, \ldots, \rho_n)$) is an imputation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that can be determined by a set of n linearly independent equations each corresponding to the equality between the characteristic value of a coalition $S \subseteq N$ and $\sum_{i\in S} x_i$, while the core conditions are satisfied for other coalitions. As Derks and Kuipers (2002) defined, the coalition whose characteristic value equals the sum of its players' allocations is called a tight coalition. Obviously, the grand coalition N is always a tight coalition. Letting $\mathcal{S}$ denote a set of n different tight coalitions that correspond to n linearly independent equations, we have

$$\boldsymbol{\rho} \in \boldsymbol{\varrho} \equiv \{\mathbf{x} | v(S) = \sum\nolimits_{i\in S} x_i,\ \forall S \in \mathcal{S} \subseteq 2^N;\ v(T) \le \sum\nolimits_{i\in T} x_i,\ \forall T \in 2^N \backslash \mathcal{S}\}, \tag{4}$$

where $\boldsymbol{\varrho}$ represents the set of extreme points. For any $\mathcal{S} \subseteq 2^N$, solving equations $\{v(S) = \sum_{i\in S} x_i$, $\forall S \in \mathcal{S}\}$ can yield a unique extreme point $\boldsymbol{\rho}$; and the extreme points for all possible different sets $\mathcal{S}$ form the set $\boldsymbol{\varrho}$.

Theorem 2 *For an ADMC game with a non-empty core, $v(N) \ge \sum_{j\in N\backslash\{i\}} (v(N) - v(N\backslash\{j\}))$, $\forall i \in N$. The core has at least one extreme point, and the maximum number of extreme points is n.*

1. *If $v(N) < \sum_{j\in N} (v(N) - v(N\backslash\{j\}))$, then there are n extreme points $\boldsymbol{\rho}_i = (\rho_i; \rho_j, j \in N\backslash\{i\})$, for $i \in N$, where $\rho_i = v(N) - \sum_{j\in N\backslash\{i\}} (v(N) - v(N\backslash\{j\}))$ and $\rho_j = v(N) - v(N\backslash\{j\})$, for $j \in N\backslash\{i\}$.*

2. *If $v(N) = \sum_{j\in N} (v(N) - v(N\backslash\{j\}))$, then there is only one extreme point in which $\rho_i = v(N) - v(N\backslash\{i\})$, for $i \in N$.*

The above theorem indicates that, for an ADMC game with a non-empty core, (i) the value of the grand coalition is no less than the sum of $n-1$ players' marginal contributions to the grand coalition, (ii) at least one and at most n extreme points exist, and (iii) the extreme points can be computed in closed form. At any extreme point, at least $n-1$ players' allocations are equal to their marginal contributions to the grand coalition.

2.2 A Closed-Form Nucleolus Solution and the Shapley Value

For an ADMC game with a non-empty core, we present a closed-form nucleolus solution, derive a sufficient condition under which the Shapley value is in the core, and compare the Shapley value and the nucleolus.

2.2.1 A Closed-Form Nucleolus Solution

Schmeidler (1969) proposed the nucleolus based on the principle of minimizing the "unhappiness" of the most unhappy coalition(s), of the second most unhappy coalition(s), etc. The unhappiness of a coalition is defined as the excess of the coalition. That is, the nucleolus solution is an n-tuple imputation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that the excess $e_S(\mathbf{x})$ of any possible coalition $S \subseteq N$ cannot be reduced without increasing any other greater excess. The excess $e_N(\mathbf{x}) = 0$ because of the collective rationality under which all players fully share the grand payoff $v(N)$. For the applications of the nucleolus in management science, see, for example, Barton (1992), Leng and Parlar (2009), and Guo, Leng, and Wang (2012).

We can use Lemma 1 and Theorem 1 to determine that the nucleolus solution of an ADMC game with a non-empty core is identical to the EANSC value (Moulin 1985), which means that each player receives his or her separable contribution and all players equally share the NSC. That is, the nucleolus solution $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is

$$y_i = (v(N) - v(N\backslash\{i\})) + \frac{\xi}{n}, \forall i \in N, \tag{5}$$

where the first term is a separable contribution of player i, and ξ is the NSC of all players, as defined in (3). Because $\xi \leq 0$, the allocation to each player can be explained as the player's marginal contribution less the per player share of the non-separable cost.

To further understand the nucleolus solution in (5) from the managerial perspective, we rewrite it as $y_i = \bar{v}_N + \eta_i, \forall i \in N$. Here, $\bar{v}_N \equiv v(N)/n$ represents the equal (average) allocation of the grand payoff $v(N)$ among n players, and $\eta_i \equiv v(N) - v(N\backslash\{i\}) - \bar{m}$ in which $\bar{m} \equiv [\sum_{j \in N}(v(N) - v(N\backslash\{j\}))]/n$ is the (overall) average marginal contribution of all players to the grand coalition. Thus, a positive, negative, or zero value of η_i corresponds to the case in which player i makes a marginal contribution above, below, or equal to the overall average. That is, under the ADMC property, when the core is non-empty, a player's payoff allocation in terms of the nucleolus solution is only dependent on all players' marginal contributions to the grand coalition. This implies that, in the case of ADMC games, Schmeidler's original "unhappiness"-based definition of nucleolus (Schmeidler 1969) could be viewed as a marginal contribution–based allocation.

When a player's marginal contribution is higher than the overall average (viz., the player is more important to the grand coalition), the player should receive a greater allocation. This can help minimize the unhappiness of each coalition. Noting that $\sum_{i \in N} \eta_i = 0$, we conclude that one or more players can gain more than the average allocation of the grand payoff whereas some or all of the others would then gain less. Moreover, the nucleolus solution in (5) is aggregate-monotonic because each player receives a higher allocation y_i ($i \in N$) if the grand payoff $v(N)$ increases. This is important, because the nucleolus may not be aggregate-monotonic even on the domain of convex games, although the aggregate monotonicity property is a desirable one for the solution concepts in cooperative game theory (Megiddo 1974 and Young 1985).

2.2.2 The Shapley Value

For an n-player coalitional game, the Shapley value for player $i \in N$ is computed as $\phi_i = \sum_{S \subseteq N \backslash \{i\}} |S|!(n-|S|-1)!(v(S \cup \{i\}) - v(S))/(n!)$, where S denotes any possible coalition excluding player i, and $|S|$ is the size of S. Although the Shapley value is a unique, monotonic solution (Megiddo 1974 and Young 1985), it may not belong to the core.

Theorem 3 *For an ADMC game with a non-empty core, if*

$$\sum_{i \in S} \frac{v(i) + (n-1)(v(N) - v(N \backslash \{i\}))}{n} \geq v(S), \text{ for } S \subset N \text{ and } |S| = n-1, \tag{6}$$

then the Shapley value is in the core. Otherwise, the Shapley value may not be in the core.

Theorem 3 indicates that to examine whether the Shapley value is the core, we can check the condition in (6) for n $(n-1)$-player coalitions. We rewrite the condition in (6) as

$$g_S \geq d_S \equiv \frac{1}{n} \sum_{i \in S} g_{\{i\}}, \text{ for } S \subset N \text{ and } |S| = n-1,$$

where, as Driessen (1985) defined, $g_S = \sum_{i \in S}(v(N) - v(N \backslash \{i\})) - v(S)$ is the gap of coalition S in the game. Because $d_S \leq \sum_{i \in S} g_{\{i\}} / |S|$, we find that, if $g_S \geq \sum_{i \in S} g_{\{i\}} / |S|$, then the condition in (6) is satisfied and the Shapley value is in the core. Note that $\sum_{i \in S} g_{\{i\}} / |S|$ is the average of the individual gaps of all players in coalition S, which is simply called the "average individual gap." Therefore, if, for any $(n-1)$-player coalition, the gap of the coalition is higher than the average individual gap, then the Shapley value belongs to the core.

Example 2 We consider the coalitional game $\mathcal{G}_1 = (N, v_1)$ in the first example of Example 1, from which we learn that $\mathcal{G}_1$ is an ADMC game with a non-empty core. Using (5), we compute the nucleolus as $\mathbf{y} = (y_1, y_2, y_3) = (4/3, 10/3, 13/3)$. The condition in (6) is satisfied, which implies that for game $\mathcal{G}_1$, the Shapley value is in the core. We compute the Shapley value as $(\phi_1, \phi_2, \phi_3) = (5/3, 19/6, 25/6)$, which satisfies all the core conditions and differs from the nucleolus $\mathbf{y}$.

Theorem 3 indicates that, if the condition in (6) does not hold for an ADMC game, then there are two possible cases: (i) the Shapley value is in the core or (ii) the Shapley value is not in the core. Next, we provide an example to illustrate case (i). Consider a coalitional game $\mathcal{G}_2 = (N, v_2)$, where $N = \{1, 2, 3\}$ and the characteristic values of all possible coalitions are given as

$$v(\varnothing) = 0;\ v(1) = 1;\ v(2) = 1;\ v(3) = 1;$$
$$v(12) = 4;\ v(13) = 6;\ v(23) = 6.5;\ v(123) = 9.$$

We find from Definition 1 and Theorem 1 that $\mathcal{G}_2$ is also an ADMC game with a non-empty core. Unlike game $\mathcal{G}_1$, the condition in (6) is not satisfied for game $\mathcal{G}_2$, because, when $S = \{1, 3\}$, $\sum_{i\in\{1,3\}}[v(i) + (n-1)(v(N) - v(N\backslash\{i\}))]/n = 17/3 < v(13) = 6$, and when $S = \{2, 3\}$, $\sum_{i\in\{2,3\}}[v(i)+(n-1)(v(N)-v(N\backslash\{i\}))]/n = 6 < v(23) = 6.5$. Thus, we cannot immediately determine whether the Shapley value is in the core. We compute the Shapley value as $(\phi_1, \phi_2, \phi_3) = (2.5, 2.75, 3.75)$, which satisfies all the core conditions and thus belongs to the core. For game $\mathcal{G}_2$, we can use the formula in (5) to obtain the nucleolus as $(y_1, y_2, y_3) = (2, 2.5, 4.5)$, which differs from the Shapley value.

We then provide another example to illustrate case (ii). Consider a coalitional game $\mathcal{G}_3 = (N, v_3)$, where $N = \{1, 2, 3\}$ and the characteristic values of all possible coalitions are given as

$$v(\varnothing) = 0;\ v(1) = 0;\ v(2) = 0;\ v(3) = 0;$$
$$v(12) = 4;\ v(13) = 6;\ v(23) = 8;\ v(123) = 9.$$

Similar to our analysis of games $\mathcal{G}_1$ and $\mathcal{G}_2$, we find that $\mathcal{G}_3$ is an ADMC game with a non-empty core; the condition in (6) is not satisfied for game $\mathcal{G}_3$, because the condition does not hold when $S = \{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. We compute the Shapley value as $(\phi_1, \phi_2, \phi_3) = (2, 3, 4)$, which does not satisfy all the core conditions, because $\phi_2 + \phi_3 = 7 < v(23) = 8$. It follows that the Shapley value is not in the core. $\triangleleft$

In Example 2, although the Shapley values in games $\mathcal{G}_1$ and $\mathcal{G}_2$ are in the core, they differ from the nucleolus. A question may arise as follows: if the Shapley value is in the core of an ADMC game with a non-empty core, which players would receive higher allocations based on the Shapley value than based on the nucleolus and would thus prefer the Shapley value for payoff allocation?

Theorem 4 *Suppose that, for an ADMC game with a non-empty core, the condition in (6) is satisfied and thus the Shapley value is in the core. If the gap of player $i \in N$ is no greater than the gap of coalition $N\backslash\{i\}$, i.e.,*

$$g_{\{i\}} \leq g_{N\backslash\{i\}}, \tag{7}$$

then player i's allocation in terms of the Shapley value is no smaller than that in terms of the nucleolus. Otherwise, player i may receive more from the nucleolus-based allocation scheme in (5) than from the Shapley value-based allocation scheme.

The above theorem exposes that if the Shapley value remains in the core, the players that satisfy the condition in (7) would prefer the Shapley value to the nucleolus, whereas the others may prefer to use the nucleolus-based allocation scheme. To illustrate our result in Theorem 4, we examine ADMC games $\mathcal{G}_1$ and $\mathcal{G}_2$ in Example 2. For game $\mathcal{G}_1$, $g_{\{1\}} = g_{\{2,3\}} = 2$, $g_{\{2\}} = 3 > g_{\{1,3\}} = 2$, and $g_{\{3\}} = 3 > g_{\{1,2\}} = 2$. As computed in Example 2, for game $\mathcal{G}_1$, $\phi_1 = 5/3 > y_1 = 4/3$; but, $\phi_2 = 19/6 < y_2 = 10/3$, and $\phi_3 = 25/6 < y_3 = 13/3$. That is, only player 1 prefers the Shapley value to the nucleolus, whereas players 2 and 3 prefer to use the nucleolus for the allocation of $v(N)$. This is consistent with our result in Theorem 4. Similarly, for game $\mathcal{G}_2$, $g_{\{1\}} = g_{\{2,3\}} = 1.5$, $g_{\{2\}} = 2 > g_{\{1,3\}} = 1.5$, and $g_{\{3\}} = 4 > g_{\{1,2\}} = 1.5$. According to Theorem 4, player 1 receives more from the Shapley value but at least one of players 2 and 3 receives more from the nucleolus. The result is consistent with our computation in Example 2: $\phi_1 = 2.5 > y_1 = 2$, $\phi_2 = 2.75 > y_2 = 2.5$, and $\phi_3 = 3.75 < y_3 = 4.5$.

2.3 The Least Core Value of an ADMC Game with an Empty Core

If the condition in (1) is not satisfied for an ADMC game, i.e., $\sum_{j \in N} v(N \backslash \{j\}) > (n-1)v(N)$, or equivalently, the NSC of all players–i.e., ξ, as given in (3)–is positive, then the core of the game is empty, and the least core and the least core value are two important solution concepts for analysis of such games. According to Shapley and Shubik (1966) and Maschler, Peleg, and Shapley (1979), the least core of a coalitional game includes all imputations that are optimal solutions to the following LP problem:

$$\varepsilon^* = \min\{\varepsilon : \textstyle\sum_{i \in N} x_i = v(N);\ e_S(\mathbf{x}) \leq \varepsilon,\ \forall S \subset N, \text{ and } S \neq \varnothing\}, \tag{8}$$

where ε^* is the least core value.

When the core is empty, the grand coalition cannot be stable because at least one coalition can gain more by leaving the grand coalition. This is likely to occur when the coalition(s) do not need to incur any cost for leaving the grand coalition. Assuming that a coalition leaving the grand coalition has to pay a "penalty" in amount of ε^*, which is the optimal solution to the LP problem in (8), we find that all coalitions can be better off if they stay in the grand coalition. This means that the grand coalition will be stable if a penalty of at least ε^* is charged. For a coalitional game with an empty core, the penalty ε^* is deemed to be the least core value, and all imputations that result in the stability of the grand coalition (under the penalty scheme) constitute the least core.

Theorem 5 *For an ADMC game with an empty core, the least core value ε^* is no smaller than the equal allocation of the NSC (i.e., ξ/n). Moreover, $\varepsilon^* = \xi/n$, if, for any player $i \in N$,*

$$\beta_i \geq \frac{\xi}{n-1}, \tag{9}$$

where $\beta_i \equiv \min\{[v(N\backslash\{k\}) - v(N\backslash\{i,k\})] - [v(N) - v(N\backslash\{i\})],\ \forall k \in N\backslash\{i\}\}$ means the minimum increase in player i's marginal contribution when he leaves the grand coalition for an $(n-1)$-player coalition.

Because of the ADMC property, player $i \in N$ can achieve an increase in his or her marginal contribution when the player leaves the grand coalition for an $(n-1)$-player coalition. Theorem 5 discloses that, if the minimum increase is no smaller than the equal allocation of the NSC among the $n-1$ players excluding player i, then the least core value reaches its lower bound ξ/n.

The least core and the least core value can also be used to analyze a coalitional game with a non-empty core, for which the least core value ε^* is non-positive, and $|\varepsilon^*|$ can be viewed as the minimum bonus that induces at least one coalition to leave the grand coalition. For an ADMC game with a non-empty core, the least core and the nucleolus coincide, and we can use (5) to calculate the least core value as $\varepsilon^* = \xi/n \leq 0$.

To illustrate the results in Theorem 5, we analyze Shapley and Shubik's "prize donation game" (1969), which is a typical example of externalities. In the game, a donor promises to distribute a prize $Z = \$3$ among three players, if and only if all players can unanimously agree with a scheme for allocating the prize among them. Otherwise, if only two players agree, then the donor distributes a smaller prize $x < Z$ between the two players, and the other player who does not agree receives nothing. In addition, there is no prize if only one player agrees. The prize donation game can be formulated as follows: $v(\varnothing) = v(i) = 0$, for $i \in N = \{1, 2, 3\}$; $v(ij) = x$, for $i, j \in N$ and $i \neq j$; and $v(N) = Z$. Shapley and Shubik (1969) provided two numerical examples with $x = \$1.5$ and $x = \$2.5$.

Using Definition 1, we find that the prize donation game is an ADMC game, if and only if $x \geq Z/2$. According to Theorem 1, when $Z/2 \leq x \leq 2Z/3$ (e.g., $x = 1.5$, as in Shapley and Shubik's first example), the game is an ADMC game with a non-empty core, and we can use the formula in (5) to compute the nucleolus solution as $y_i = Z/3$, $\forall i \in N$. When $2Z/3 < x < Z$ (e.g., $x = 2.5$, as in Shapley and Shubik's second example), it is an ADMC game with an empty core, and we can use Theorem 5 to compute the least core value. Because $\beta_i = 2x - Z$ (for any player $i \in N$) and $\xi/(n-1) = (3x - 2Z)/2$, the condition in (9) is satisfied and the least core value can be computed as $\varepsilon^* = (3x - 2Z)/3$. That is, if any coalition who disagrees with the allocation of prize Z and leaves the grand coalition has to pay a "penalty" in amount of $(3x - 2Z)/3$, then every player will remain in the grand coalition and agree with the prize allocation.

3 Applications in Management Science

In practice, the search for economies of scale and the reduction of risk are two of the major motivations for firms to form coalitions or alliances (Porter and Fuller 1986). The economies of scale and risk pooling could give rise to the ADMC property.

3.1 Application in the Airline and Shipping Areas: Analysis of a Code-Sharing Game

Strategic alliances are a prevalent form of cooperation between two or among three or more business entities in network-oriented industries such as air transport, shipping, telecommunications, multimodal transportation, and logistics industries (Zhang and Zhang 2006). As a common practice in the airline industry (Oum et al. 2002) and the ocean shipping industry (Sheppard and Seidman 2001), horizontal cooperation among multiple firms can result in freight cost savings for these firms and is viewed as an efficient solution to the improvement of freight transportation (Ergun, Kuyzu and Savelsbergh 2007). In the airline industry, carriers (e.g., Northwest, Continental, and Delta) cooperate extensively via code sharing, joint frequent flyer programs, and strategic alliances (e.g., Star Alliance, SkyTeam, and Oneworld) to expand their networks, improve revenues, reduce costs, and increase customer benefits (Iatrou and Alamdari 2005).

Under a code-sharing program, two or more airlines share a single flight to increase capacity utilization and service frequency. However, the marginal contribution made by an airline could decrease with the number of airlines in the program. Thus, we may construct and analyze an ADMC game to find the allocation of cost savings generated by horizontal cooperation among multiple airlines. This research issue is important because the lack of a fair mechanism to allocate the cost savings is one of the major impediments for multi-firm cooperation in logistics (Cruijssen, Cools and Dullaert 2007). For the airline industry, the study of effective revenue-sharing mechanisms is extremely limited, although there is a rich literature on strategic alliances (Çetiner 2013).

Next, we develop a code-sharing game in characteristic function form, in which three or more airlines ($n \geq 3$) share a single flight under a codeshare agreement. We learn from Wensveen (2007) and from Vasigh, Fleming, and Tacker (2008) that for a flight operation, an air carrier incurs a fixed cost and a variable cost. The fixed cost consists of direct operating (operations-related) costs that do not vary with changes in available seat-miles (ASMs). Here, a seat-mile is one passenger seat transported one statute mile. The typical direct operating costs include the costs of the flight (e.g., flight crew expenses, fuel and oil, airport and en route charges, etc.), maintenance and overhaul costs, and depreciation and amortization costs. The variable cost is composed of indirect operating (passenger-related) costs that depend mainly on the ASMs. Such indirect costs include station and ground expenses, passenger service costs, and others.

According to the above, an airline coalition $S \subseteq N$ incurs the fixed cost Π and the variable cost $\pi(S)$ that is dependent on the total ASMs in S. For coalition S, the cost savings can be computed as the sum of the airlines' operating costs before joining the coalition minus the total operating cost of the coalition and can be treated as its characteristic value. That is, $v(S) = \sum_{i \in S}(\Pi + \pi(i)) - (\Pi + \pi(S)) = \Pi(|S| - 1) + \sum_{i \in S} \pi(i) - \pi(S)$.

Proposition 1 *The code-sharing game is an ADMC game, if* $\Pi \geq \Pi_0 \equiv \max\{\pi(N) - \pi(N \backslash \{i\}) - \pi(i), \forall i \in N\}$*, and the variable cost* $\pi(\cdot)$ *is a convex function.*

Proposition 1 discloses that a code-sharing game with a sufficiently large fixed cost and a convex variable cost is an ADMC game. Such a game is common in practice because of the following two facts. First, as de Arantes Gomes Eller and Moreira (2014) mentioned, the fixed costs represent approximately 65% of an airline's total cost. This means that in the airline industry, the fixed cost is usually very large. Second, according to the findings of Vasigh, Fleming, and Tacker (2008), the marginal cost could be increasing, which implies that the variable cost could be a convex function. In a number of relevant papers (e.g., Barbot 2004), the variable cost is assumed to be a linear function of the ASMs. Under such an assumption, the two sufficient conditions in Proposition 1 are met because $\pi(T \cup \{i\}) - \pi(T) - \pi(i) = 0$, and $\pi(T \cup \{i\}) - \pi(T) = \pi(S \cup \{i\}) - \pi(S)$, for player $i \in N$ and for coalitions S and T such that $S \subseteq T \subseteq N \backslash \{i\}$. Thus, the game is an ADMC game.

Corollary 1 *The code-sharing game has a non-empty core, if it is an ADMC game and* $\Pi \geq (n - 1)\pi(N) - \sum_{j \in N} \pi(N \backslash \{j\})$.

Corollary 1 reveals that the code-sharing game is an ADMC game with a non-empty core if the fixed cost is sufficiently large. This can occur in reality, because, as mentioned above, the fixed cost is usually very large in the airline industry. When the core of the code-sharing game is non-empty, we can use the formula in (5) to compute the nucleolus for the game.

Moreover, the code-sharing game is an ADMC game with an empty core, if the variable cost $\pi(\cdot)$ is a convex function and the fixed cost Π is given such that $\Pi_0 \leq \Pi < (n-1)\pi(N) - \sum_{j \in N} \pi(N \backslash \{j\})$. Using Theorem 5, we find that the least core value $\varepsilon^* \geq [(n-1)\pi(N) - \sum_{j \in N} \pi(N\backslash\{j\}) - \Pi]/n$, where the equality holds if $\Pi \geq (n-1)\{\pi(N\backslash\{i\}) + \pi(N\backslash\{k\}) - \pi(N\backslash\{i,k\})\} - \sum_{j \in N} \pi(N\backslash\{j\})$, for $i, k \in N$ and $i \neq k$.

3.2 Application in the Purchasing Area: Analysis of a Group-Buying Game

Group buying (which is a kind of cooperative purchasing) is a common practice in many professional fields such as grocery, health care, electronics, industrial manufacturing, and agriculture. Two or more firms form a purchasing group or consortium (coalition) to obtain a price discount on goods or services from vendors based on the aggregate purchasing quantity (Chen and Roma 2011). The price discount obtained by a purchasing group is usually increasing but concave in the purchase quantity (Schotanus, Telgen, and de Boer 2008). Therefore, the cost savings that a firm can contribute to a larger purchasing group is smaller, which indicates that the ADMC property may appear in group buying. Thus, we can construct an ADMC game to determine the allocation of cost savings among the members in a purchasing group.

We analyze a group-buying game in characteristic function form, in which $n \geq 3$ retailers jointly purchase a product from a supplier to achieve cost savings. As considered by Schotanus, Telgen, and de Boer (2009) and by Chen and Roma (2011), the supplier charges the retailers a quantity-dependent unit wholesale price for the product. The wholesale price is modeled as a continuous quantity discount function (QDF) $w(q) = a + d/q^e$, where $a > 0$ is the base price, $d > 0$ is the discount scale, q is the purchase quantity, and e is the steepness. Schotanus, Telgen, and de Boer (2009) showed that the QDF with $e \in [-1, 1.6]$ fits well with 66 discount schedules in practice.

Retailer $i \in N$ purchases $q_i > 0$ units of the product from the supplier. If the retailer uses individual purchasing, its total purchase cost is $q_i w(q_i) = aq_i + d(q_i)^{1-e}$. However, if the retailers in coalition $S \subseteq N$ use group purchasing, they will enjoy a lower wholesale price $w(q_S) = a + d/(q_S)^e$, where $q_S \equiv \sum_{i \in S} q_i$. As a result, in coalition S, retailer i's total purchase cost is $q_i w(q_S) = aq_i + dq_i/(q_S)^e$. By joining the group-buying coalition S, retailer i can achieve the cost savings as $q_i w(q_i) - q_i w(q_S) = d[(q_i)^{1-e} - q_i(q_S)^{-e}]$. Thus, we can compute the characteristic value of coalition S (i.e., the total cost savings realized by all retailers in S) as

$$v(S) = \sum_{i \in S}[q_i w(q_i) - q_i w(q_S)] = d\left[\sum_{i \in S}(q_i)^{1-e} - (q_S)^{1-e}\right].$$

Proposition 2 *The group-buying game is an ADMC game if q^{1-e} is a decreasing, convex function. That is, the group-buying game with $e \geq 1$ is an ADMC game.*

The above proposition indicates that the property of function q^{1-e} is important to determine whether the group-buying game is an ADMC game. As Schotanus, Telgen, and de Boer (2009) showed, in practice, the value of steepness e falls in the range $[-1, 1.6]$. We find that if e is in the range $[1, 1.6]$, then the function q^{1-e} is decreasing and convex. We can thus conclude that the group-buying problem can be viewed as an ADMC game, when the steepness for the quantity discount is sufficiently large such that $e \in [1, 1.6]$.

Corollary 2 *For every group-buying game that is an ADMC game, the core is non-empty.*

We learn from Corollary 2 that if a group-buying game is an ADMC game (i.e., $e \geq 1$), then its core is non-empty. This could occur in practice, when steepness e is in the range $[1, 1.6]$.

3.3 Application in the Manufacturing Area: Analysis of a Scheduling Profit Game

Schulz and Uhan (2013) considered a submodular profit game $\mathcal{G} = (N, v)$ that involves three or more agents ($n \geq 3$). Each agent has a job with a unit processing time (viz., the processing time of each job is 1) and a deadline $d_i \in \mathbb{Z}_{>0}$. This implies that if a single agent processes her job on a machine, then there is no tardiness for the job (if $d_i \geq 1$). If the agent's job is completed by its deadline, the agent can earn a profit $w_i \in \mathbb{R}_{\geq 0}$; otherwise, the agent obtains nothing. All agents in a coalition $S \subseteq N$ schedule their jobs on a single machine to maximize the total profit, which is the characteristic value of the coalition $v(S)$. The machine can process only one job at a time. To have an incentive to join the grand coalition N, all agents need a scheme for allocating the total profit $v(N)$ among them. We simply call this game a "scheduling profit game." As Schulz and Uhan (2013) argued, game $\mathcal{G}$ is a submodular profit game with an empty core.

We consider a variation of the above game, denoted by $\mathcal{G}' = (N, v')$, which differs from Schulz and Uhan's game $\mathcal{G}$ (2013) in the following two aspects. First, when a coalition uses the machine to process its agents' jobs, it incurs a machine setup cost $s > 0$ before the jobs are processed. Second, if the job of an agent $i \in N$ is completed after its deadline, the agent still gains a profit w_i but also incurs a tardiness loss $t \in (0, \min\{w_i, i \in N\}]$. In reality, the machine setup cost may not be negligible, and an agent may still obtain a positive profit despite a loss due to job tardiness. Therefore, in game $\mathcal{G}'$, for any coalition $S \subseteq N$ and $S \neq \varnothing$, $v'(S) = \sum_{i \in S \setminus L_S} w_i + \sum_{i \in L_S} (w_i - t) - s$, where L_S denotes the set of delayed jobs under the optimal schedule for coalition S.

Proposition 3 *The scheduling profit game $\mathcal{G}'$ with machine setup cost s and job tardiness loss t is an ADMC game when $t \leq s$.*

When we consider a machine setup cost and a job tardiness loss in scheduling, the resulting profit game $\mathcal{G}'$ is an ADMC game if the job tardiness loss is no more than the setup cost. Next, to illustrate the application of our result in Theorem 1, we use the inequality in (1) to derive a sufficient condition for the non-emptiness of the core of $\mathcal{G}'$.

Corollary 3 *The core of the scheduling profit game $\mathcal{G}'$ is non-empty if $t \leq \min\{s, (\sum_{j \in N} w_j - v'(N))/|L_N|\}$.*

When the scheduling profit game $\mathcal{G}'$ has a non-empty core, the formula in (5) can be used to compute the nucleolus for the game. Moreover, if $(\sum_{j \in N} w_j - v'(N))/|L_N| < t \leq s$, the game is an ADMC game with an empty core, and Theorem 5 can be used to find the lower bound of the least core value.

4 Summary and Concluding Remarks

We consider multi-player allocation problems under the ADMC property, for which we develop ADMC games and derive a necessary and sufficient condition for the non-emptiness of the core. The nucleolus solution of an ADMC game with a non-empty core is the EANSC value. For an n-player ADMC game with a non-empty core, we obtain closed-form solutions for the extreme points of the core, and find that at least one extreme point exists and that the maximum number of extreme points is n. We also derive a sufficient condition under which the Shapley value remains in the core. Moreover, a player can receive a higher allocation based on the Shapley value in the core than based on the nucleolus, if the gap of the player is no greater than the gap of the $(n-1)$-player complementary coalition. For the ADMC games with an empty core, although it may be difficult to compute the least core value, we derive its lower bound and obtain a closed-form solution under a condition. Both the lower bound and the closed-form solution are obtained as the equal allocation of the NSC. We analyze a prize donation game (Shapley and Shubik 1969) to illustrate our results.

To demonstrate the applicability of ADMC games in management science, we analyze three games: a code-sharing game, a group-buying game, and a scheduling profit game. We show that a code-sharing game with a sufficiently large fixed cost and a convex variable cost function, a group-buying game with a greater-than-one steepness in quantity discount, and a scheduling profit game with a machine setup cost and a sufficiently small job tardiness loss are all ADMC games. Such situations could occur in practice, as indicated by relevant empirical or analytical findings in some publications.

In summary, this paper contributes to the literature by (i) introducing the class of ADMC games, (ii) deriving a necessary and sufficient condition for the non-emptiness of the core of an ADMC game, (iii) computing the extreme points of the core and discussing the Shapley value for ADMC games, and (iv) presenting various applications in management science. We expect that the new class of games and our results can help improve the application of cooperative game theory in management science.

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Appendix A Proofs

Proof of Lemma 1. We consider an imputation $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in the non-empty core. For an $(n-1)$-player coalition $T = N \backslash \{i\}$ $(i \in N)$, the excess $e_T(x) = v(T) - v(N) + x_i$ is non-positive because $v(N) = \sum_{j \in N} x_j = \sum_{j \in T} x_j + x_i \geq v(T) + x_i$. For any coalition $S \subseteq T$ and $S \neq \varnothing$, under the ADMC property, we have

$$\begin{aligned} e_{S\cup\{i\}}(\mathbf{x}) - e_S(\mathbf{x}) &= v(S \cup \{i\}) - v(S) - x_i \\ &\geq v(T \cup \{i\}) - v(T) - x_i \\ &= v(N) - v(N\backslash\{i\}) - x_i, \end{aligned}$$

which is non-negative according to our above argument for an $(n-1)$-player coalition. Therefore, we find the first property that $e_S(\mathbf{x}) \leq e_{S\cup\{i\}}(\mathbf{x})$, $\forall i \in N \backslash S$ and $1 \leq |S| \leq n-1$.

According to the above result, whenever the n-tuple imputation $\mathbf{x}$ is in the core, the largest excess is the excess of an $(n-1)$-player coalition. The number of $(n-1)$-player coalitions is n. We assume that among the n coalitions, the excess of coalition I is the largest and the excess of coalition J is the smallest. That is, $e_I(\mathbf{x}) = \max\{e_M(\mathbf{x}), \forall M \subseteq N, |M| = n-1\}$ and $e_J(\mathbf{x}) = \min\{e_M(\mathbf{x}), \forall M \subseteq N, |M| = n-1\}$, for $I, J \subseteq N$, $I \neq J$, and $|I| = |J| = n-1$. Without loss of generality, we let $I = N\backslash\{i\}$ and $J = N\backslash\{j\}$, where $i, j \in N$. If $e_I(\mathbf{x}) > e_J(\mathbf{x})$, then, to reduce the largest excess $e_I(\mathbf{x})$, we can increase the value of x_j and decrease the value of x_i. As a result, the excess $e_I(\mathbf{x})$ is reduced but the excess $e_J(\mathbf{x})$ is increased, and other excesses are not changed. When $e_I(\mathbf{x}) = e_J(\mathbf{x})$, we need not further change the values of x_i and x_j, because, otherwise, $e_J(\mathbf{x}) > e_I(\mathbf{x})$. Then, we can similarly reduce the second largest excess(es), the third largest excess(es), and others until the excesses of all $(n-1)$-player coalitions are equal. ■

Proof of Theorem 1. We first show the sufficiency of the condition. For an $(n-1)$-player coalition $S = N\backslash\{i\}$ $(i \in N)$, we compute the excess $e_S(\mathbf{x})$ as $e_S(\mathbf{x}) = v(S) - \sum_{j \in S} x_j = v(N\backslash\{i\}) - v(N) + x_i$. Considering an imputation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $x_i = (v(N) + \sum_{j \in N} v(N\backslash\{j\}))/n - v(N\backslash\{i\})$, for $i \in N$, we can rewrite $e_S(\mathbf{x})$ as $e_S(\mathbf{x}) = (v(N) + \sum_{j \in N} v(N\backslash\{j\}))/n - v(N)$, which is non-positive if the condition in (1) is satisfied.

For an $(n-2)$-player coalition $S = N\backslash\{i, j\}$ $(i, j \in N)$, we compute the difference between the excesses $e_{S\cup\{i\}}(\mathbf{x})$ and $e_S(\mathbf{x})$ as follows: $e_{S\cup\{i\}}(\mathbf{x}) - e_S(\mathbf{x}) = v(S \cup \{i\}) - v(S) - x_i \geq v(S \cup \{i, j\}) - v(S \cup \{j\}) - x_i$, where the inequality follows the ADMC property. Letting $T \equiv S \cup \{j\}$ (which is an $(n-1)$-player coalition), we find that $v(S \cup \{i, j\}) - v(S \cup \{j\}) - x_i = v(N) - v(T) - x_i$, which is non-negative as shown in our above argument for the $(n-1)$-player coalition. It thus follows that $e_{S\cup\{i\}}(\mathbf{x}) - e_S(\mathbf{x}) \geq 0$, and the excesses of all $(n-2)$-player coalitions are non-positive. Similarly, we can show by induction that for the imputation $\mathbf{x}$, all excesses are non-positive, i.e., $e_S(\mathbf{x}) \leq 0$, $\forall S \subseteq N$. That is, the imputation $\mathbf{x}$ is in the core, which means that the core is non-empty.

We now prove the necessity of the condition. When the core is non-empty, at least one feasible solution exists for the following LP problem: $\min x_1$ subject to (i) $e_S(\mathbf{x}) \leq 0$, $\forall S \subseteq N$ and (ii) $\sum_{i \in N} x_i = v(N)$, because each feasible solution corresponds to an imputation in the non-empty core. Next, we find a feasible solution to the above LP problem, which is thus in the core.

We learn from the first result in Lemma 1 that for every imputation $\mathbf{x}$ in the core, the largest excess is the excess of an $(n-1)$-player coalition. This means that if the excesses of all $(n-1)$-player coalitions are non-positive, then the excesses of all coalitions are non-positive. That is, in the LP problem, if, $\forall S \subseteq N$ such that $|S| = n-1$, constraint (i) is satisfied, then the constraint is satisfied for all $S \subseteq N$. Therefore, we need only examine the excesses of all $(n-1)$-player coalitions.

As the second result in Lemma 1 indicates, the largest excess reaches its minimum if the excesses of all $(n-1)$-player coalitions are equal. This implies that if an imputation makes the excesses of all $(n-1)$-player coalitions equal, then it is a feasible solution to the above LP problem, because any other imputation raises the largest excess and thus makes constraint (i) more difficult to satisfy. Accordingly, we solve the equations $e_S(\mathbf{x}) = e_T(\mathbf{x})$, $\forall S, T \subseteq N$, where $S \neq T$, and $|S| = |T| = n-1$, which can be written as

$$\begin{cases} x_i - x_1 = v(N\backslash\{1\}) - v(N\backslash\{i\}), \ \forall i \in N\backslash\{1\}, \\ \sum_{j \in N} x_j = v(N). \end{cases} \tag{10}$$

Solving the equations in (10) yields the unique solution $x_i = (v(N) + \sum_{j\in N} v(N\backslash\{j\}))/n - v(N\backslash\{i\})$, $\forall i \in N$, which is in the non-empty core. For this imputation, $e_S(\mathbf{x}) =(\sum_{j\in N} v(N\backslash\{j\}) - (n-1)v(N))/n \leq 0$; this implies that the condition in (1) is satisfied when the core is non-empty. That is, the condition is also a necessary condition for the non-emptiness of the core. ■

Proof of Theorem 2. We begin by discussing which coalitions are tight if there exists an extreme point $\boldsymbol{\rho} \in \boldsymbol{\varrho}$. Peleg and Sudhölter (2007) noted that extreme points can be determined based on minimal balanced collections (MBCs). According to the Bondareva-Shapley theorem (Bondareva 1963 and Shapley 1967), if and only if the core of a coalitional game is non-empty, then, for each MBC $\mathcal{B}$,

$$\textstyle\sum_{S\in\mathcal{B}} \delta_S v(S) \leq v(N), \tag{11}$$

where $\{\delta_S\}_{S\in\mathcal{B}}$ is the system of balancing weights for $\mathcal{B}$ such that $\sum_{S\in\mathcal{B}:j\in S} \delta_S = 1$, for every $j \in N$. We learn from Theorem 1 that the inequality in (1) is the necessary and sufficient condition for the non-emptiness of an ADMC game, which is the same as the condition in (11) for the MBC $\mathcal{B} = \{N\backslash\{i\}\}_{i\in N}$ with the balancing weights $\delta_{N\backslash\{i\}} = 1/(n-1)$. That is, for an ADMC game, when the condition in (1) (i.e., the condition in (11) for the MBC consisting of all the $(n-1)$-player coalitions) is satisfied, the core is non-empty, which implies that the conditions in (11) for the other MBCs are also satisfied. As a result, for an ADMC game with a non-empty core, we need only consider the MBC consisting of n $(n-1)$-player coalitions. Note that the grand coalition N is always a tight coalition, i.e., $\sum_{i\in N} x_i = v(N)$. For an n-player ADMC game with a non-empty core, because a set of n linearly independent equations is needed to uniquely calculate an extreme point, the existence of an extreme point requires that the minimum number of $(n-1)$-player tight coalitions is $n-1$. Therefore, for any ADMC game with a non-empty core, the maximum number of extreme points is n.

We learn from Theorem 1 that $\sum_{j\in N} (v(N) - v(N\backslash\{j\})) \geq v(N)$. Before computing the extreme points, we investigate whether it is possible for an ADMC game with a non-empty core to possess the condition $\sum_{j\in N\backslash\{i\}} (v(N) - v(N\backslash\{j\})) > v(N)$, for $i \in N$. Letting $N\backslash\{l\} = \{j_1, j_2, \ldots, j_{n-2}, j_{n-1}\}$ (where $l \in N$ and $j_{n-1} = i$) and $S_k \equiv N\backslash\{l, j_1, \ldots, j_k\}$ $(k = 1, 2, \ldots, n-1)$, we have

$$\begin{aligned} v(N) &= v(N) - v(N\backslash\{l\}) + (v(N\backslash\{l\}) - v(S_1)) + (v(S_1) - v(S_2)) \\ &\quad + \ldots + (v(S_{n-3}) - v(S_{n-2})) + v(S_{n-2}), \end{aligned}$$

where $S_{n-2} = \{j_{n-1}\} = \{i\}$. According to the ADMC property in Definition 1, we find that $v(N\backslash\{l\}) - v(S_1) \geq v(N) - v(N\backslash\{j_1\})$ and $v(S_{t-1}) - v(S_t) \geq v(N) - v(N\backslash\{j_t\})$, for $t = 2, \ldots, n-2$. Therefore,

$$v(N) \geq \textstyle\sum_{j\in N\backslash\{i\}}(v(N) - v(N\backslash\{j\})) + v(i). \tag{12}$$

Assuming that the condition $\sum_{j\in N\backslash\{i\}}(v(N) - v(N\backslash\{j\})) > v(N)$ $(i \in N)$ is satisfied, we can rewrite the inequality in (12) as

$$v(N) > v(N) + v(i). \tag{13}$$

Because the inequality in (13) cannot be satisfied because $v(i) \geq 0$, we conclude that the condition $\sum_{j\in N\backslash\{i\}} (v(N) - v(N\backslash\{j\})) > v(N)$ cannot hold for any ADMC game. Then, we consider the following two scenarios.

1. $v(N) < \sum_{j\in N}(v(N) - v(N\backslash\{j\}))$. In this case, each extreme point corresponds to a set of $(n-1)$-player tight coalitions (i.e., $N\backslash\{j\}$, $j \in N\backslash\{i\}$) and thus, there are n extreme points. For the set of coalitions $N\backslash\{j\}$, $j \in N\backslash\{i\}$, the extreme point can be obtained as $\rho_j = v(N) - v(N\backslash\{j\})$, $j \in N\backslash\{i\}$, and $\rho_i = v(N) - \sum_{j\in N\backslash\{i\}}(v(N) - v(N\backslash\{j\}))$, $i \in N$.

2. $v(N) = \sum_{j\in N}(v(N) - v(N\backslash\{j\}))$. For this case, all the $(n-1)$-player coalitions are tight. Therefore, there are $n+1$ tight coalitions (including the grand coalition). For any tight coalition, the corresponding equation is linearly dependent on n linearly independent equations that result from the other tight coalitions. Thereby, there is only one extreme point in which $\rho_i = v(N) - v(N\backslash\{i\})$, $\forall i \in N$.

Summarizing the above, we find that, for an n-player ADMC game with a non-empty core, the core has at least one extreme point, and the maximum number of extreme points is n. ■

Proof of Theorem 3. Using Definition 1, we find that, for any ADMC game, $v(S\cup\{i\}) - v(S) \geq v(N) - v(N\backslash\{i\})$, for $S \subseteq N\backslash\{i\}$ and $|S| \geq 1$. Thus, the Shapley value for player $i \in N$ is computed as:

$$\begin{aligned}
\phi_i &= \frac{\sum_{S\subseteq N\backslash\{i\}}|S|!(n-|S|-1)!(v(S\cup\{i\})-v(S))}{n!} \\
&\geq \frac{v(i)}{n} + (v(N)-v(N\backslash\{i\}))\frac{\sum_{S\subseteq N\backslash\{i\},|S|\geq 1}|S|!(n-|S|-1)!}{n!} \\
&= \frac{v(i)}{n} + (v(N)-v(N\backslash\{i\}))\frac{(n-1)(n-1)!}{n!} \\
&= \frac{v(i)+(n-1)(v(N)-v(N\backslash\{i\}))}{n}.
\end{aligned}$$

Therefore, the Shapley value is in the core, if

$$\sum_{i\in S}\frac{v(i)+(n-1)(v(N)-v(N\backslash\{i\}))}{n} \geq v(S), \text{ for } S \subset N \text{ and } |S| \geq 1. \tag{14}$$

Suppose that for a non-empty coalition $S \subset N$, the inequality in (14) is satisfied. We re-write this inequality as

$$\begin{aligned}
&\sum_{i\in S\backslash\{j\}}\frac{v(i)+(n-1)(v(N)-v(N\backslash\{i\}))}{n} + \frac{v(j)+(n-1)(v(N)-v(N\backslash\{j\}))}{n} \\
\geq\ & v(S) - v(S\backslash\{j\}) + v(S\backslash\{j\}),
\end{aligned} \tag{15}$$

where $j \in S$. According to Definition 1, we have

$$\begin{aligned}
\frac{v(j)+(n-1)(v(N)-v(N\backslash\{j\}))}{n} &= v(N)-v(N\backslash\{j\}) + \frac{v(j)+v(N\backslash\{j\})-v(N)}{n} \\
&\leq v(N)-v(N\backslash\{j\}) \\
&\leq v(S)-v(S\backslash\{j\}).
\end{aligned} \tag{16}$$

It follows from the inequalities in (15) and (16) that

$$\sum_{i\in S\backslash\{j\}} \frac{v(i)+(n-1)(v(N)-v(N\backslash\{i\}))}{n} \geq v(S\backslash\{j\}).$$

We can thus conclude that if the inequality in (14) is satisfied for $S \subset N$, then it is also satisfied for $S\backslash\{j\}$ $(\forall j \in S)$. This result implies that, to examine whether the Shapley value is in the core, we need not test the inequality in (14) for all possible coalitions, but only to check it for all $(n-1)$-player coalitions. Hence, we have the condition as in (6).

However, when the condition in (6) is not satisfied, we cannot decide on whether the Shapley value belongs to the core. If an $(n-1)$-player coalition S exists such that each player $i \in S$ makes a sufficiently small marginal contribution to any possible coalition including this player, then the sum of the Shapley values for all players in S (i.e., $\sum_{i\in S}\phi_i$) could be smaller than $v(S)$. Therefore, the Shapley value may not be in the core, if the condition in (6) is not satisfied. ■

Proof of Theorem 4. We learn from the proof of Theorem 3 that the Shapley value for player $i \in N$ satisfies the following inequality:

$$\phi_i \geq v(N) - v(N\backslash\{i\}) - \varsigma_1/n, \text{ where } \varsigma_1 \equiv v(N) - v(N\backslash\{i\}) - v(i).$$

Recall from our discussion of the nucleolus solution in (5) that the nucleolus for player $i \in N$ can be rewritten as

$$y_i = v(N) - v(N\backslash\{i\}) - \varsigma_2/n, \text{ where } \varsigma_2 \equiv \sum_{j\in N}(v(N) - v(N\backslash\{j\})) - v(N).$$

To compare the nucleolus y_i and the Shapley value ϕ_i (for $i \in N$), we compute

$$\varsigma_1 - \varsigma_2 = (v(N) - v(i)) - \sum_{j\in N\backslash\{i\}}(v(N) - v(N\backslash\{j\})) = g_{\{i\}} - g_{N\backslash\{i\}},$$

where $g_{\{i\}} = v(N) - v(N\backslash\{i\}) - v(i)$ is the gap of player i and $g_{N\backslash\{i\}} = \sum_{j\in N\backslash\{i\}}(v(N) - v(N\backslash\{j\})) - v(N\backslash\{i\})$ is the gap of coalition $N\backslash\{i\}$. If $g_{\{i\}} \leq g_{N\backslash\{i\}}$, then $\varsigma_1 \leq \varsigma_2$ and $\phi_i \geq y_i$.

Otherwise, if $g_{\{i\}} > g_{N\backslash\{i\}}$, then $\varsigma_1 > \varsigma_2$; but we cannot find if ϕ_i is larger or smaller than y_i. Letting

$$\phi_i = v(N) - v(N\backslash\{i\}) - \varsigma_1/n + \mu, \text{ where } \mu \geq 0,$$

we have

$$y_i - \phi_i = (\varsigma_1 - \varsigma_2)/n - \mu,$$

which is positive if μ is sufficiently small such that $n\mu < \varsigma_1 - \varsigma_2$. That is, if $g_{\{i\}} > g_{N\backslash\{i\}}$, then player i may receive more from the nucleolus–based allocation scheme in (5) than from the Shapley value–based allocation scheme. ■

Proof of Theorem 5. According to Theorem 1, we find that the core of an ADMC game is empty if and only if $\sum_{j\in N} v(N\backslash\{j\}) > (n-1)v(N)$. For an imputation $\mathbf{x} = \{x_i, i \in N\}$ such that $v(N) = \sum_{i\in N} x_i$, we can rewrite the above inequality as $\sum_{j\in N} v(N\backslash\{j\}) > (n-1)\sum_{i\in N} x_i$, or,

$\sum_{j\in N}[v(N\backslash\{j\}) - \sum_{i\in N\backslash\{j\}} x_i] = \sum_{j\in N} e_{N\backslash\{j\}}(\mathbf{x}) > 0$. This means that if the core of an ADMC game is empty, then player $j \in N$ exists such that $e_{N\backslash\{j\}}(\mathbf{x}) > 0$. Hence, the least core value ε^* is larger than or equal to the minimum value of $\varepsilon_1 \equiv \max\{e_{N\backslash\{j\}}(\mathbf{x}), j \in N\}$. The value of ε_1 is minimized when $e_{N\backslash\{j\}}(\mathbf{x}) = e_{N\backslash\{i\}}(\mathbf{x})$, $\forall i, j \in N$ and $i \neq j$. That is, when $x_i = x_i^* \equiv [v(N) + \sum_{j\in N} v(N\backslash\{j\})]/n - v(N\backslash\{i\})$, for $i \in N$, the minimum value of ε_1 is $\varepsilon_1^* = \xi/n$.

To compute the least core value when the condition in (9) is satisfied, we rewrite the condition as

$$\sum_{j\in N} v(N\backslash\{j\}) \leq (n-1)[v(N\backslash\{i\}) + v(N\backslash\{k\}) - v(N\backslash\{i,k\})],\ \forall i, k \in N \text{ and } i \neq k, \tag{17}$$

and consider another coalitional game $\mathcal{G}' = (N, v')$ such that $v'(N) = v(N) + n\varepsilon_1^*/(n-1)$ and $v'(S) = v(S)$, $\forall S \subset N$. According to Definition 1, game $\mathcal{G}'$ is an ADMC game, if, for $i, j \in N$ and $i \neq j$, $v'(N) - v(N\backslash\{j\}) \leq v(N\backslash\{i\}) - v(N\backslash\{i,j\})$, which is equivalent to the condition in (17). It addition, it is easy to see that $\bar{\mathbf{x}}' \equiv \{\bar{x}_i' = x_i^* + \varepsilon_1^*/(n-1), i \in N\}$ is an imputation that satisfies the equation $v'(N) = \sum_{i\in N} x_i'$. Moreover, for $j \in N$, $e'_{N\backslash\{j\}}(\bar{\mathbf{x}}') = v'(N\backslash\{j\}) - \sum_{i\in N\backslash\{j\}}[x_i^* + \varepsilon_1^*/(n-1)] = e_{N\backslash\{j\}}(\mathbf{x}^*) - \varepsilon_1^* = 0$, where $\mathbf{x}^* = \{x_i = x_i^*, i \in N\}$. Thus, $\sum_{j\in N} v'(N\backslash\{j\}) = (n-1)\sum_{i\in N} \bar{x}_i' = (n-1)v'(N)$, which indicates that the condition in Theorem 1 is satisfied.

Therefore, if the condition in (17) is satisfied, then game $\mathcal{G}'$ has a non-empty core, viz., the set $\{\mathbf{x}' : \sum_{i\in N} x_i' = v'(N);\ e'_S(\mathbf{x}') \leq 0, \forall S \subset N$, and $S \neq \varnothing\}$ is non-empty. Because the non-empty set includes element $\bar{\mathbf{x}}'$ and $e'_S(\bar{\mathbf{x}}')$ is increasing in $|S|$ (as shown by Lemma 1), the ε_1^*-core of game $\mathcal{G}$—i.e., $\{\mathbf{x} : \sum_{i\in N} x_i = v(N);\ e_S(\mathbf{x}) \leq \varepsilon_1^*, \forall S \subset N$, and $S \neq \varnothing\}$—contains element $\mathbf{x}^*$; thus, it is non-empty. Because the least core value ε^* is no smaller than ε_1^*, we find that $\varepsilon^* = \varepsilon_1^*$. ■

Proof of Proposition 1. For this proof, we should consider the condition in Definition 1 for the code-sharing game. For all $i \in N$, $S \subseteq T \subseteq N\backslash\{i\}$, and $|S| \geq 1$, $v(S\cup\{i\}) - v(S) = \Pi - \pi(S\cup\{i\}) + \pi(S) + \pi(i)$ and $v(T\cup\{i\}) - v(T) = \Pi - \pi(T\cup\{i\}) + \pi(T) + \pi(i)$. Therefore,

$$[v(S\cup\{i\}) - v(S)] - [v(T\cup\{i\}) - v(T)] = \pi(T\cup\{i\}) - \pi(S\cup\{i\}) + \pi(S) - \pi(T),$$

which is non-negative when $\pi(T\cup\{i\}) - \pi(T) \geq \pi(S\cup\{i\}) - \pi(S)$—viz., the variable cost $\pi(\cdot)$ is a convex function.

Next, we consider the condition that $v(T\cup\{i\}) \geq v(T) + v(i) = v(T)$, $\forall i \in N$, $\forall T \subseteq N\backslash\{i\}$, which is satisfied when $\Pi \geq \pi(N) - \pi(N\backslash\{i\}) - \pi(i)$, $\forall i \in N$, because of the ADMC property. The above indicates that if the fixed cost Π is sufficiently large, then the condition is satisfied. ■

Proof of Corollary 1. Because $v(S) = \Pi(|S|-1) + \sum_{i\in S}\pi(i) - \pi(S)$, we can use (1) to obtain the condition for non-emptiness of the core as $\Pi \geq (n-1)\pi(N) - \sum_{j\in N}\pi(N\backslash\{j\})$. ■

Proof of Proposition 2. We consider the condition in Definition 1 for the group-buying game. For all $i \in N$, $S \subseteq T \subseteq N\backslash\{i\}$, and $|S| \geq 1$,

$$[v(S\cup\{i\}) - v(S)] - [v(T\cup\{i\}) - v(T)] = d[(q_S)^{1-e} - (q_T)^{1-e} + (q_T + q_i)^{1-e} - (q_S + q_i)^{1-e}],$$

which is non-negative when $(q_T + q_i)^{1-e} - (q_T)^{1-e} \geq (q_S + q_i)^{1-e} - (q_S)^{1-e}$, viz., the function q^{1-e} is a convex function (i.e., $\partial^2 q^{1-e}/\partial q^2 \geq 0$). Differentiating q^{1-e} twice w.r.t. q yields $\partial^2 q^{1-e}/\partial q^2 =$

$e \times q^{-1-e} \times (e-1)$, which is positive when $e \leq 0$ or $e \geq 1$.

In addition, in ADMC games, $v(T \cup \{i\}) \geq v(T) + v(i) = v(T)$, $\forall i \in N$, $\forall T \subseteq N \backslash \{i\}$, which is satisfied when $v(N) \geq v(N \backslash \{i\})$, $\forall i \in N$, because of the ADMC property. That is, the characteristic value of a coalition $N \backslash \{i\}$ is no larger than that of the grand coalition N, which is satisfied when $e \geq 1$. ■

Proof of Corollary 2. In the group-buying game, $v(S) = d[\sum_{i \in S} (q_i)^{1-e} - (q_S)^{1-e}]$. When the condition in Proposition 2 (i.e., $e \geq 1$) is satisfied, the group-buying game is an ADMC game. To ensure that the condition in (1) is satisfied, we require that $\sum_{j \in N} (q_{N \backslash \{j\}})^{1-e} \geq (n-1)(q_N)^{1-e}$, which holds when q^{1-e} is a decreasing function, or, $e \geq 1$. ■

Proof of Proposition 3. We examine the conditions in Definition 1 for game $\mathcal{G}'$. For $i \in N$, we consider $S \subseteq T \subseteq N \backslash \{i\}$ and $|S| \geq 1$. There are two possibilities for agent i's job under the optimal schedule for coalition $S \cup \{i\}$.

1. Agent i's job is delayed after agent i joins coalition S (i.e., $i \in L_{S \cup \{i\}}$). Then, agent i's job is also delayed after agent i joins coalition T (i.e., $i \in L_{T \cup \{i\}}$). Thus, $v'(S \cup \{i\}) - v'(S) = v'(T \cup \{i\}) - v'(T) = w_i - t$.

2. Agent i's job is not delayed after agent i joins coalition S (i.e., $i \notin L_{S \cup \{i\}}$). This occurs in two possible cases. In the first case, $L_{S \cup \{i\}} = L_S$, and $v'(S \cup \{i\}) - v'(S) = w_i$. In the second case, an agent $j \in S$ exists such that $j \notin L_S$ but $j \in L_{S \cup \{i\}}$, and we find that $v'(S \cup \{i\}) - v'(S) = w_i - t$. Because $i \notin L_{S \cup \{i\}}$, similar to our above discussion for coalition S, when agent i joins coalition T, this agent's job may or may not be delayed.

   (a) If agent i's job is delayed (i.e., $i \in L_{T \cup \{i\}}$), then $v'(T \cup \{i\}) - v'(T) = w_i - t$.

   (b) If agent i's job is not delayed (i.e., $i \notin L_{T \cup \{i\}}$), then there are two possible cases. In the first case, $L_{T \cup \{i\}} = L_T$, $v'(T \cup \{i\}) - v'(T) = w_i$. In the second case, an agent $k \in T$ exists such that $k \notin L_T$ but $k \in L_{T \cup \{i\}}$, and we have $v'(T \cup \{i\}) - v'(T) = w_i - t$.

   Because $L_{S \cup \{i\}} \subseteq L_{T \cup \{i\}}$, the condition $v'(S \cup \{i\}) - v'(S) \geq v'(T \cup \{i\}) - v'(T)$ holds.

Moreover, we require that, for any $i \in N$ and $T \subseteq N \backslash \{i\}$, $v'(T \cup \{i\}) \geq v'(T) + v'(i)$. There are two possibilities for agent i's job under the optimal schedule for coalition $T \cup \{i\}$.

1. Agent i's job is delayed after agent i joins T (i.e., $i \in L_{T \cup \{i\}}$). We find that $v'(T \cup \{i\}) = v'(T) + w_i - t$. As $v'(T) + v'(i) = v'(T) + w_i - s$, the inequality $v'(T \cup \{i\}) \geq v'(T) + v'(i)$ holds if $t \leq s$.

2. Agent i's job is not delayed after agent i joins coalition T (i.e., $i \notin L_{T \cup \{i\}}$). Similar to our above discussions, there are two possible cases. In the first case, $L_{T \cup \{i\}} = L_T$, and $v'(T \cup \{i\}) = v'(T) + w_i \geq v'(T) + w_i - s = v'(T) + v'(i)$. In the second case, an agent $j \in T$ exists such that $j \notin L_T$ but $j \in L_{T \cup \{i\}}$. In this case, $v'(T \cup \{i\}) = v'(T) + w_i - t$, and the inequality $v'(T \cup \{i\}) \geq v'(T) + v'(i)$ holds if $t \leq s$.

Summarizing the above, we find that $v'(T \cup \{i\}) \geq v'(T) + v'(i)$ if $t \leq s$. ■

Proof of Corollary 3. We learn from Proposition 3 that game $\mathcal{G}'$ is an ADMC game if $t \leq s$. Next, we investigate the condition in (1). There are two possible results when agent j joins the grand coalition N.

1. If agent j's job is delayed (i.e., $j \in L_N$), then $v'(N) = v'(N \backslash \{j\}) + w_j - t$, which indicates $v'(N \backslash \{j\}) = v'(N) - (w_j - t)$.

2. If agent j's job is not delayed (i.e., $j \notin L_N$), then, similar to our discussion in the proof of Proposition 3, there are two possible cases. In the first case, $L_N = L_{N \backslash \{j\}}$, and $v'(N) = v'(N \backslash \{j\}) + w_j$, which indicates $v'(N \backslash \{j\}) = v'(N) - w_j$. In the second case, an agent $k \in N \backslash \{j\}$ exists such that $k \notin L_{N \backslash \{j\}}$ but $k \in L_N$, and we find that $v'(N) = v'(N \backslash \{j\}) + w_j - t$, or, $v'(N \backslash \{j\}) = v'(N) - (w_j - t)$.

Based on the above results, the condition in (1) is found as $\sum_{j \in N \backslash L_N} (v'(N) - w_j) + \sum_{j \in L_N} [v'(N) - (w_j - t)] \leq (n-1)v'(N)$, or, $t \leq (\sum_{j \in N} w_j - v'(N))/|L_N|$. Hence, we obtain the condition as shown in this corollary. ■